THE EXACT CORRECTION THEOREM FOR SPACES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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In the present paper, for a space $BV(X)$ of functions of generalized bounded variation, constructed on the base of a symmetric space $X$ of sequences, we investigate the problem of correcting functions on sets of arbitrarily small measure by functions of a Lipschitz space. In the case where $X = l^p$ and, consequently, $BV(l^p)$ is a space of functions of bounded $p$-variation, this problem was solved by Ryazanov [1]. In the case where $X$ is an Orlicz space $l_h$ and, consequently, $BV(l_h)$ is a space of functions of bounded variation in the Wiener–Young sense, and in the case where $X$ is a Lorentz space $\Lambda(\varphi)$ and, consequently, $BV(\Lambda(\varphi))$ is a space of functions of bounded variation in Waterman’s sense [2, 3], this problem was studied by the author [4]. In this paper we prove a general theorem that shows the available possibilities of correcting functions for the problem considered. Namely, we show that these possibilities are defined completely by existing mutual embeddings of $X$ and the corresponding Orlicz space. We establish a correction test for correcting functions of $BV(X)$ by functions of a Lipschitz space $\text{Lip}_w$; by this correction test, any function of $BV(X)$ can be corrected on a set of arbitrarily small measure by a function of $\text{Lip}_h^{-1}$ if and only if the symmetric space $X$ can be embedded into the Orlicz space $l_h$. Based on this test and the embedding theorem, we characterize the Orlicz spaces of sequences in terms of the correction and embedding of functions of generalized bounded variation. We note that the results obtained imply that if a symmetric space $X$ is not an Orlicz space, then there is always a “gap” between the Lipschitz space embedded in $BV(X)$ and the Lipschitz space of functions, corrected on a set of arbitrarily small measure, by which we correct functions of $BV(X)$, i.e., these Lipschitz spaces do not coincide. This possibility appears to be first noted in [4] in connection with the study of the problem for spaces of functions of generalized bounded variation in Waterman’s sense.

Let us recall some results of the theory of symmetric spaces of sequences. These properties are quite analogous to the properties of symmetric spaces of functions discussed in detail in [5, 6].

Let $\{e_i\}_{i=1}^{\infty}$ be a standard basis in a numerical sequence space. A sequence space $X$ is called symmetric if for any sequence $\{\lambda_i : |\lambda_i| \leq 1\}$ and for any element $\sum a_i e_i \in X$ we have $\sum a_i \lambda_i e_i \in X$ and

$$\left\| \sum a_i \lambda_i e_i \right\|_X \leq \left\| \sum a_i e_i \right\|_X,$$

$$\left\| \sum a_i e_i \right\|_X = \left\| \sum a_i^* e_i \right\|_X,$$

here $\{a_i^*\}$ is the permutation of the sequence $\{|a_i|\}$ in nonascending order.

We give some commonly used examples of symmetric spaces of sequences.

Let $\{\varphi_i\}_{i=1}^{\infty}$ be a positive concave sequence, i.e., $2\varphi_i \geq \varphi_{i-1} + \varphi_{i+1}$ and $\varphi_0 = 0$. A Lorentz space $\Lambda(\varphi)$ (Martinskevich space $M(\varphi)$) consists of sequences $\sum a_i e_i$ such that the norm

$$\left\| \sum a_i e_i \right\|_{\Lambda(\varphi)} = \sum_{i=1}^{\infty} a_i^*(\varphi_i - \varphi_{i-1})$$

(respectively, $\left\| \sum a_i e_i \right\|_{M(\varphi)} = \sup_{i \geq 1} \frac{\varphi_i}{i} \sum_{i=1}^{\infty} a_i^*$)

is finite.

Let $h(t)$ be an $N$-function [7, p. 149]. An Orlicz sequence space $l_h$ consists of sequences such that the norm

$$\left\| \sum a_i e_i \right\|_{l_h} = \inf \left\{ \lambda > 0 : \sum_{i=1}^{\infty} h(a_i/\lambda) \leq 1 \right\}$$
is finite. In the definition of Orlich sequence spaces, the behavior of \( h(t) \) in a neighborhood of zero is essential; for this reason, in what follows we sometimes define \( h(t) \) only in a neighborhood of zero, completing its definition to some \( N \)-function on \( R \) in some way.

Let \( J = [0, 1] \). Below we assume that all functions are defined on \( J \). As usual, we denote the continuity module on \( J \) by \( w \), and the corresponding Lipschitz space by \( \text{Lip} w \). By [8, p. 78], \( w \) can be considered as a concave increasing function.

The following definition is the most important in the present paper.

**Definition 1.** Let \( X \) be a symmetric space of sequences. The space \( BV(X) \) of functions of generalized bounded variation is the set of functions \( f : J \rightarrow R \) such that the norm

\[
\|f|BV(X)\| = \sup_{\{J_i\}} \left\| \sum f(J_i) |X| + |f(0)| \right\|
\]

is finite; here \( \{J_i\} \) denotes the set of pairwise nonoverlapping intervals of \( J \), and \( f(J_i) := f(b_i) - f(a_i) \) for \( J_i = (a_i, b_i) \).

Let us illustrate this definition by examples. If \( X \) is the Orlich space \( l_h \), then \( BV(l_h) \) is simply the space of functions of bounded \( h \)-variation in the Wiener–Young sense. If \( X \) is a Lorentz space \( \Lambda(\varphi) \), then \( BV(\Lambda(\varphi)) \) is the space of functions of generalized bounded variation in Waterman’s sense [2, 3]. If \( X \) is a Martsinkevich space \( M(\varphi) \), then \( BV(M(\varphi)) \) is the space of functions of generalized bounded variation in Chanturiya’s sense [9]; this space was called by Chanturiya the space with bounded variation module.

**Definition 2.** Let \( X \) be a symmetric space of sequences, and let \( BV(X) \) be the corresponding space of functions of generalized bounded variation. We say that the space \( BV(X) \) is corrected by the Lipschitz space \( \text{Lip} w \) if for an arbitrary \( s > 0 \) and for a function \( f \in BV(X) \) there exists a function \( f_\varepsilon \in \text{Lip} w \) such that \( \mu\{t : f(t) \neq f_\varepsilon(t)\} < \varepsilon \).

The main result of this paper is given by the following theorem.

**Theorem 1.** Let \( X \) be a symmetric space, and let \( BV(X) \) be a space of functions of generalized bounded variation. Then \( BV(X) \) is corrected by a Lipschitz space \( \text{Lip} h^{-1} \) if and only if the following continuous embedding holds:

\[
X \subset l_h.
\]

The proof of this theorem is based on the following statements.

**Theorem 2.** Let \( X \) be a symmetric space of sequences, and let \( BV(X) \) be the corresponding space of functions of generalized bounded variation. Suppose that \( h(t) \) is an \( N \)-function. If \( X \) is continuously embedded in the Orlich space \( l_h \), then \( BV(X) \) is corrected by the Lipschitz space \( \text{Lip} h^{-1} \).

It is sufficient to prove the theorem for the case \( X = l_h \). Suppose that \( f \in BV(l_h) \) and \( \|f|BV(l_h)\| \leq 1 \). We fix \( \varepsilon > 0 \) and put

\[
g(t) = \sup_{\{J_i\}} \left\| \sum f(J_i) e^t |l_h| \right\|
\]

where \( \{J_i\} \) is a sequence of pairwise nonoverlapping intervals of \([0, t]\). Then the function \( g(t) \) is monotone and \( g(1) \leq 1 \). By the correction theorem for the space \( V_1 \) of functions of bounded variation [10, p. 57], there exists a perfect set \( M \) of measure greater than \( (1 - \varepsilon) \) such that

\[
|g(t) - g(s)| \leq C(\varepsilon)|t - s|, \quad t, s \in M.
\]

Therefore, for \( t, s \in M \) we have

\[
h(|f(t) - f(s)|) \leq |g(t) - g(s)| \leq C(\varepsilon)|t - s|
\]

or

\[
|f(t) - f(s)| \leq h^{-1}(C(\varepsilon)|t - s|) \leq \max\{1, C(\varepsilon)\} h^{-1}(|t - s|).
\]

We put \( f_\varepsilon(t) = f(t), t \in M \), and continue \( f_\varepsilon \) to the whole interval \( J \) by linearity. Then \( f_\varepsilon \in \text{Lip} h^{-1} \) and \( \mu\{t : f_\varepsilon(t) \neq f(t)\} < \varepsilon \). The theorem has been proved.

Going back to Theorem 1, we need some auxiliary constructions to prove the necessity condition.

We need the following simple lemma on the density point for a set of positive measure.

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