OPERATIONAL REPRESENTATION FOR THE LAGUERRE POLYNOMIALS

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(Presented by P. Turán)

1. In 1960, Carlitz [1] gave the following operational formula involving Laguerre polynomials:

\[ \prod_{j=1}^{n} (xD - x + \alpha + j) = n! \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x) D^p \]

where

\[ L_{n}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x}). \]

Later, Chatterjea [2] developed certain operational formulas involving Laguerre and Bessel polynomials in order to derive already known formulas or to obtain new ones. For the Laguerre polynomials, we may note the following formulas of Chatterjea:

\[ \Omega_{n}Y = \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x) D^p Y, \]

where

\[ \Omega_{n} = \sum_{p=0}^{n} \left( \frac{\alpha+n}{n-p} \right) \frac{x^p}{p!} (D - 1)^p; \]

\[ \frac{x^n}{n!} \left[ D + \frac{\alpha+n-x}{x} \right]^{n} Y = \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x) D^p Y. \]

Recently Al-Salam [3] has given the following operational formula for the Laguerre polynomials:

\[ [x(1+xD)]^{n} (x^\alpha e^{-x} f(x)) = x^{\alpha+n} e^{-x} n! \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x) D^p f(x), \]

wherefrom he has obtained the following operational representation of the Laguerre polynomials:

\[ [x(1+xD)]^{n} (x^\alpha e^{-x}) = x^{\alpha+n} e^{-x} n! L_{n}^{(\alpha)}(x). \]

In [3, p. 127], Al-Salam has mentioned the following operational formula of A. M. Chak:

\[ L_{n}^{(\alpha)}(x) = x^{-\alpha-n-1} e^{x} (x^2 D)^{n} (x^{2} e^{-x}). \]
The object of this note is to give simple proofs of the formulas due to CARLITZ, AL-SALAM and CHAK. Also we shall show that the formulas of AL-SALAM and CHAK are special cases of the following formula:

\[(x(\delta - k + 1))^n(x^\alpha e^{-x} Y) = n! x^{\alpha+k+n} e^{-x} \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x) D^p Y\]  

(1.8) (where \(\delta \equiv xD\)), which we propose to prove in the present note. We shall also point out several consequences of (1.8).

2. For our purpose, we shall require the following well-known properties of the operator \(\delta\):

\[(2.1) F(\delta)[x^\alpha f(x)] = x^\alpha F(\alpha + x) f(x)\]
\[(2.2) F(\delta)[e^{\theta(x)} f(x)] = e^{\theta(x)} F(\delta + x g') f(x)\]
\[(2.3) x^\alpha F(\delta) F(\delta + \alpha) \ldots F(\delta + (n-1)\alpha) = [x^\alpha F(\delta)]^n.\]

Now we observe

\[x^{-\alpha} e^x D^n(x^{\alpha+n} e^{-x} Y) = x^{-\alpha-n} e^x \delta (\delta-1) \ldots (\delta-n+1)(x^{\alpha+n} e^{-x} Y) =\]
\[= e^x(\delta + \alpha + n)(\delta + \alpha + n - 1) \ldots (\delta + \alpha + 1)(e^{-x} Y) =\]
\[= (\delta - x + \alpha + n)(\delta - x + \alpha + n - 1) \ldots (\delta - x + \alpha + 1) Y = \prod_{j=1}^{n} (\delta - x + \alpha + j) Y.\]

Thus we have proved that

\[(2.4) D^n(x^{\alpha+n} e^{-x} Y) = x^\alpha e^{-x} \prod_{j=1}^{n} (\delta - x + \alpha + j) Y\]

which was proved by CARLITZ with the help of induction. Next CARLITZ proved by Leibniz theorem

\[(2.5) D^n(x^{\alpha+n} e^{-x} Y) = n! x^\alpha e^{-x} \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x) D^p Y.\]

Thus (1.1) follows immediately from (2.4) and (2.5).

To derive AL-SALAM's formula (1.5), we first note

\[(2.6) \prod_{j=1}^{n} (xD - x + \alpha + j) f(x) = e^x \prod_{j=1}^{n} (xD + \alpha + j) e^{-x} f(x) =\]
\[= x^{-\alpha} e^x \prod_{j=1}^{n} (\delta + j) x^\alpha e^{-x} f(x) = x^{-\alpha} e^x x^{-n}[x(\delta + 1)]^n(\alpha^\alpha e^{-x} f(x))\]

(where \(\delta \equiv xD\)). Thus AL-SALAM's formula (1.5) follows from (1.1) and (2.6).