STRESSES IN A HOLLOW ROTATING CYLINDRICALLY ORTHOTROPIC TUBE

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Explicit stress distributions for a hollow cylindrically orthotropic tube due to spinning are presented for immediate use in everyday engineering work. The solutions are obtained directly without using the stress function concept. Various boundary condition combinations present in practical applications are considered. Examples are given in order to indicate how the degree of anisotropy influences the stress distributions.

1. Introduction

The high strength to weight and stiffness to weight ratios of composite materials have increased their potential use in applications where the rotational speeds are increasing. Various kinds of shafts, rolls and roll covers in paper mills, and cores of paper reels are typical examples. Preliminary calculations concerning the global dynamic behavior can be easily done by using handbook solutions [1] whenever equivalent modulus of elasticity in the longitudinal direction has been calculated using laminate theory [2] or commercial software for point analysis.

As the motivation to use composite materials is often the increase in rotational speed, the stress distributions due to centrifugal forces in a cross section become sometimes interesting for the end user, and, in some applications, they can be the cause of the failure due to the lower strength in the direction perpendicular to the fibers. The difficulty in practical engineering design is that handbooks do not contain ready to use equations for calculating such stresses.

The theory of elasticity of cylindrically anisotropic solids has been comprehensively studied, for example, by Lekhnitskii [3]. The presentation contains as an example the stress distributions in a hollow, cylindrically anisotropic tube under internal and external pressure. Using the basic theory given in the book, the stress function may also be solved in the cases under consideration here. It appears, however, that a much easier way is to solve the stress distributions directly from the basic equations, viz., using the equation of equilibrium of the stress components and the compatibility equation. As a result, we have one second order differential equation for the radial stress component instead of a fourth degree differential equation obtained for the stress function. After solving the radial stress distribution, the circumferential stress component distribution is obtained directly from the radial equilibrium condition. Another advantage of this approach is that the very typical displacement boundary condition (prescribed radial displacement when, for example, the composite material is wound over a stiff steel core) can be very easily incorporated into the formulation.

The results are given in explicit form. The correctness of the results has been confirmed by comparing them, in special cases, to the results available in literature. All the derivations have been checked also with the symbolic manipulation program DERIVE [4]. Parametric studies are made in a dimensionless form to show the effect of the relative values of the elastic coefficients in different directions on the type of stress distribution.

2. Theory

2.1. Statements, Restrictions and Nomenclature. A hollow, rotating cylinder or disk made of an orthotropic material is considered. The analysis includes various axisymmetric boundary conditions. Plane strain conditions are studied.
thoroughly, and the plane stress is obtained as a special case. The radial and circumferential (hoop) stress fields are the primary unknowns. The goal is a nondimensional presentation with least possible number of parameters. In what follows, \( r \), \( \theta \), and \( x \) mean the radial, circumferential, and axial coordinates, respectively. It is assumed that material principal axes coincide with these coordinate directions. The notation used, for example, by Jones [2] is accepted for the Poisson’s ratios. Thus, the first and the second indexes mean the direction of stress and contraction (strain), accordingly. The material is assumed to be macroscopically homogeneous, its mass density being \( \rho \), and the cylinder is rotating around the \( x \)-axis with angular velocity \( \omega \). The inner radius of the cylinder is \( a \) and the outer one is \( b \).

2.2. General Solution for the Stress Components. From the radial equilibrium of a differential piece \( dr \times r \theta \), we obtain

\[
\sigma_r + \sigma_\theta - \sigma_\phi = -\rho \omega^2 \rho \rho^2,
\]

where the prime means differentiation with respect to \( r \). By eliminating the radial displacement from the well-known definitions for the radial and circumferential strains, we get the compatibility condition

\[
\varepsilon_r - \varepsilon_\theta + \sigma_\phi = 0.
\]

The strains in the radial and circumferential directions are obtained using the generalized Hooke’s law for an orthotropic material (see, for example, [3]) and letting the axial strain to be zero. The result is

\[
\varepsilon_r = (1 - \nu_{r\theta}) \frac{\sigma_r}{E_r} - (\nu_{r\theta} + \nu_{r\phi}) \frac{\sigma_\theta}{E_\theta};
\]

\[
\varepsilon_\theta = -(\nu_{r\theta} + \nu_{r\phi}) \frac{\sigma_r}{E_r} + (1 - \nu_{r\phi}) \frac{\sigma_\phi}{E_\phi}.
\]

Note that the cross-sectional components of strains for a plane stress condition are obtained from (3) by letting all the Poisson’s ratios having index \( x \) be zero. The same argument holds for all subsequent results. Next, we substitute the strains from Eq. (3) to Eq. (2) and use Eq. (1) for \( \sigma_\phi \). By using the symmetry properties of the compliance terms, we obtain a nondimensional differential equation for the radial stress. It reads

\[
\zeta \sigma_r'' + \zeta \sigma_r' + (1 - \gamma) \sigma_r = -(3 + \beta) \zeta^4
\]

where \( \zeta = r/b \), and the nondimensional stress is defined as

\[
\tilde{\sigma}_r = \frac{\sigma_r}{\rho \omega^2 \rho^2}.
\]

A stationary cylinder is not considered here. In that case, the differential equation for the radial stress would be homogeneous, but otherwise the derivations are analogous if we choose some other scaling factor for the stresses.

In Eq. (4), a prime means differentiation with respect to \( \zeta \). The nondimensional parameters \( \gamma \) and \( \beta \) depend on the material constants. Some of the various forms for them are

\[
\gamma = \frac{E_\theta}{E_r} \frac{1 - \nu_{r\theta}^2}{E_\theta} \frac{E_r}{E_r} = \frac{E_\theta}{E_r} \frac{1 - \nu_{r\theta}^2}{E_\theta} \frac{E_r}{E_r} = \frac{E_\theta}{E_r} \frac{1 - \nu_{r\theta}^2}{E_\theta} \frac{E_r}{E_r} > 0
\]

and

\[
\beta = \frac{\nu_{r\theta} + \nu_{r\phi} \nu_{r\theta}}{1 - \nu_{r\theta}^2} \frac{E_\theta}{E_r} \frac{1 - \nu_{r\theta}^2}{E_\theta} \frac{E_r}{E_r} = \frac{\nu_{r\theta} + \nu_{r\phi} \nu_{r\theta}}{1 - \nu_{r\theta}^2} \frac{E_\theta}{E_r} \frac{1 - \nu_{r\theta}^2}{E_\theta} \frac{E_r}{E_r}.
\]

The second forms for \( \gamma \) and \( \beta \) are perhaps most useful and informative. The solutions for the radial and circumferential stresses depend on the material properties through only two nondimensional parameters. To calculate these parameters, we must know the values for three Poisson’s ratios and two ratios between the extensional moduli. The positiveness of