A CLASS OF NONDISPERSE EVOLUTION EQUATIONS
WITH SOLITARY WAVE SOLUTIONS

A. ABBAS, A.C. BRYAN and A.E.G. STUART
Department of Mathematics, The City University, Northampton Square,
London EC1V 0HB, England

ABSTRACT. We present a class of nonlinear evolution equations possessing stable solitary wave
solutions with a sech² profile. These equations are related to the Korteweg–de Vries (KdV) and
regularised longwave (RLW) equations, but, unlike the latter, are dispersion free in the linear limit.

A feature of certain conservative, nonlinear evolution equations is the interaction between non-
linearity and dispersion that allows the existence of solitary wave solutions which, in some cases,
turn out to be solitons as well. For example, in the Korteweg–de Vries (KdV) and regularised
longwave (RLW) equations

KdV: $u_t + u_x + uu_x + u_{xxx} = 0,$
(1)

RLW: $u_t + u_x + uu_x - u_{xxt} = 0,$
(2)

the nonlinear term $uu_x$ counteracts the dispersion introduced by the third-order terms $u_{xxx}$ and
$u_{xxt}$ respectively, leading in each case to the existence of solitary wave solutions with those for
the KdV forming solitons. The RLW was suggested as an alternative to the KdV as a model equation
for nonlinear unidirectional waves by Benjamin et al. [1] because of its physically preferable
dispersion relation and regular behaviour. Other alternatives have been suggested by Joseph and
Egri [2].

Quite apart from the demands of a particular model for specific dispersion relations, it seems
to be generally believed that both dispersion and nonlinearity are necessary for conservative
evolution equations to have stable solitary wave solutions (see, for example, Scott et al. [3]). In
this paper we show that this belief is unfounded by presenting counterexamples.

We begin by considering the class of equations

$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} = 0,$
(3)

where $u(x, t)$ is a real scalar field for all $(x, t) \in \mathbb{R}^2$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Equation (3) reduces
to the KdV and RLW equations when $(a_1, a_2, a_3, a_4) = (1, 0, 1, 0)$ and $(1, 0, 0, -1)$ respectively.
The action of the linear terms in (3) on $a(k) \exp (i(kx - \omega(k)t))$ leads to the dispersion relation

Copyright © 1982 by D. Reidel Publishing Company
Thus, when \( a_3 = a_4 \), the class (3) is nondispersive.

Next we show that these nondispersive equations have solitary wave solutions with a sech\(^2\) profile. Transforming the evolution equation to the rest frame of the solitary wave via the substitutions \( x \to \xi = x - (1 + c)t \), \( c \in \mathbb{R} \), \( t \to t \) and \( u(x, t) \to u(\xi, t) \) gives

\[
u_t - cu_x + \alpha uu_x + a_2 uu_{\xi\xi} + \beta cu_{\xi\xi\xi} - \beta u_{\xi\xi\xi} = 0,\]

where \( \alpha = (a_1 - (1 + c)a_2)/c \) and \( \beta = -a_3 = -a_4 \). Thus, the solitary wave, if it exists, is a solution of the ordinary differential equation

\[-u_{\xi\xi} + a uu_{\xi} + \beta u_{\xi\xi\xi} = 0.\]

The general solution of (6) is a Weierstrassian elliptic function (see, for example, Hancock [4]) which can be written in terms of a Jacobian elliptic cosine amplitude of modulus \( k \) to give

\[u(\xi) \approx \text{cn}^2 (\lambda \xi, k),\]

where \( \lambda \) and \( k \) depend upon \( \alpha \) and the integration constants. Since \( \text{cn} (\lambda \xi, k) \) is periodic for all \( k^2 \neq 1 \), the solitary wave forms can only occur when \( k^2 = 1 \) and, for general \( \alpha \), this requires the boundary conditions \( u, u_{\xi}, u_{\xi\xi} \to 0 \) as \( |\xi| \to \infty \). With these boundary conditions the solitary wave solutions of the original nondispersive equation (3) are

\[u_s(\xi, t) = (3/\alpha) \text{sech}^2 ((x - (1 + c)t)/2\sqrt{\beta}),\]

provided that \(-a_3 = -a_4 = \beta > 0\). For positive amplitudes we also require \( \alpha > 0 \). Note that the above also shows that these equations possess elliptic travelling wave solutions which is a property that they share with the KdV.

To investigate the stability of these solitary wave solutions (7), consider solutions \( \nu(\xi, t) \) of (5) (i.e., the evolution equation in the rest frame of \( u_s \)) with initial profiles which are perturbations of the solitary wave. Then, for linear stability, we can write

\[\nu(\xi, t) = u_s(\xi) + ef(\xi, t) + o(e^2),\]

where \( f \) has the differentiability properties required by (5) and \( f(\xi, 0) \to 0 \) as \( |\xi| \to \infty \). Now substituting (8) into (5), using the fact that \( u_s(\xi) \) satisfies (6) and working to first order in \( e \), gives the following equation for \( f(\xi, t) \)

\[f_t - cf_x + \alpha c(u_s f_x + u_{s\xi} f_\xi) + a_2 u_{s\xi} f_x + \beta c f_{\xi\xi\xi} - \beta f_{\xi\xi\xi} = 0,\]

where \( u_{s\xi} = \frac{du_s}{d\xi} \). Since \( f \) satisfies a linear equation we can express it as the superposition

\[f(\xi, t) = \sum \phi(\xi, \omega) e^{i\omega t} + \int \phi(\xi, \Omega) e^{i\nu t} d\Omega,\]

where the summation is over the discrete spectrum of (9) with \( \omega = 0 \) and \( \omega = \mu + iv, (v \neq 0) \), and