A CONSTRUCTION OF THE JET OF THE RESOLVENT

F. GUIL

Departamento de Métodos Matemáticos de la Física, Facultad de Ciencias Físicas,
Universidad Complutense, Madrid-3, Spain

ABSTRACT. We consider an equation for the diagonal of the resolvent of matrix differential operators which appears associated to a general Riccati equation. We show that the power series solutions of these equations determines the jet of the resolvent used by Gel'fand and Dikii in the theory of Lax equations. The variational derivatives of such solutions are also calculated.

0. INTRODUCTION

The characterization of an infinite collection of Lax equations, i.e., equations of the form
\[ L_t = LP - PL \]
for a differential operator \( L = \sum_{k=0}^{\infty} u_k D^k \) with matrix coefficients \( u_k \), was given in [1] where differential operators \( P \) of an arbitrary order which pair with \( L \), were found in terms of the jet of the nucleus of the resolvent operator (the quantities \( S_k \) of Section 2). Actually, the symbols of such operators \( P \) are the coefficients of the expansion in negative power of \( z \) of the series \( \sum_{k=0}^{\infty} S_{-k-1}(z) z^k \).

This paper deals with the problem of solving the systems of relations (2.1), (2.2) deduced in [1] for the collection \( \{S_k\}_{k=-m}^{\infty} \). The method followed here, essentially generalizes to the matrix case the techniques presented in [2] for scalar operators. In the present matrix case this method applies except for first-order differential operators, but for first-order operators the equations of the jet can be directly handled without difficulty.

To define the jet \( \{S_k\}_{k=-\infty}^{\infty} \) we analyze the power series solutions of two equations: the Riccati equation (1.2) and an equation for the diagonal of the resolvent (2.3). This strange-looking equation is motivated by the problem of calculating the variational derivatives of the solutions of the Riccati equation, given in Section 3. Besides the construction of the \( S_k \) (definition (2.5)) we find a relation between the trace of the diagonal of the resolvent and the trace of the solutions \( \chi \) of the Riccati equation. This relation tells us that the coefficients of \( \text{tr} \chi \) are also first-integrals for the Lax equations since this was proved for the coefficients of the trace of the resolvent [1].

Finally, it is of interest to compare this resolvent method with the approaches presented in [3] and [4].

1. THE MATRIX RICCATI EQUATION

We consider the differential ring \( A \) of polynomials in \( u_0, u_1, u_2, \ldots, u_{n-1}, u_n \) and their derivatives.
$u_{k,\alpha\beta}^{(0)} = D^{l} u_{k,\alpha\beta}, \alpha, \beta = 1, 2, ..., m; j = 0, 1, ...$ The derivation $D$ acts on $A$ according to the formula $Df = \sum u_{k,\alpha\beta} \frac{\partial f}{\partial u_{k,\alpha\beta}}$. We denote by $A_{m}$ the ring of $m \times m$ matrices with entries on $A$ and by $A_{m}[z^{-1}]$ the ring of formal Laurent power series of the form $\sum_{r=0}^{\infty} f_{r} z^{-r}$, $f_{r} \in A_{m}$, $r_{0}$ an arbitrary integer. The derivation $D$ acts on $A_{m}[z^{-1}]$ term by term and is defined on $A_{m}$ by $D(f_{0}) = (Df_{0})$.

For a given $\kappa \in A_{m}[z^{-1}]$ we introduce the differential operator $\tau$

$$\tau a = Da + ak, \quad a \in A_{m}[z^{-1}]$$

(1.1)

to define the matrix differential polynomials in $\kappa$, $P_{k}(\kappa) = \tau^{k}E$, $k = 0, 1, ...$ ($E$ denotes the $m \times m$ identity matrix) which extends to the matrix case the scalar $P_{k}$ used in [2]. One finds $P_{0} = E$, $P_{1} = \kappa$, $P_{2} = \kappa^{1} + \kappa^{2}$, $P_{3} = \kappa^{1} + \kappa^{2} + 2\kappa^{1} + \kappa^{3}$, ... and the recurrence formula $P_{k+1} = \tau P_{k}$ which follows from the definition of the $P_{k}$.

On $A_{m}[z^{-1}]$ we examine the solutions $\chi$ of the equation

$$\sum_{k=0}^{n} u_{k}P_{k}(\kappa) = z^{n}E$$

(1.2)

where $u_{n}$ is an invertible diagonal matrix $u_{n} = \text{diag}(c_{1}, ..., c_{m})$, the $c_{\alpha}$ being non-zero constants and the entries of the $u_{k}$ free generators of the ring $A$, $k = 0, 1, ..., n - 1$.

The Riccati-type equation (1.2) is satisfied by a series of the form

$$\chi = k z + \sum_{r \geq 0} \chi_{r}z^{-r}, \quad \chi_{r} \in A_{m}$$

(1.3)

$k$ being a constant diagonal matrix such that $k^{n} = u_{n}^{-1}$. By direct substitution of (1.3) in (1.2) we get

$$\left(u_{n} \sum_{l=0}^{n-1} k^{l} \alpha k^{n-1-l} + u_{n-1} k^{n-1}\right) z^{n-1} + R_{n-2}(\alpha)z^{n-2} + ... + R_{0}(\alpha) = 0$$

where $\alpha = \sum_{r \geq 0} \chi_{r}z^{-r}$ and the $R_{0}, R_{1}, ..., R_{n-2}$ are certain matrix polynomials in $\alpha, \alpha', ..., \alpha^{(n-1)}$. This formula shows us that the matrix coefficients of $\alpha^{*} = \Sigma k^{l} \alpha k^{n-1-l}$ can be calculated by recurrence only for those 'roots' $k$ which make the map $\alpha \to \alpha^{*}$ bijective as we shall suppose here (see, for example the discussion given in [3] and [4]). That $\chi_{r} \in A_{m}$ is a direct consequence of the formula written above for calculating $\alpha$.

2. AN EQUATION FOR THE DIAGONAL OF THE RESOLVENT

In [1] were deduced the relations

$$\sum_{k=0}^{n} \sum_{\alpha=0}^{k} \binom{k}{\alpha} u_{k} S_{k+l-\alpha}^{(\alpha)} = z^{n}S_{l}$$

(2.1)