Interval Analytic Treatment of Convex Programming Problems

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Abstract — Zusammenfassung

Interval Analytic Treatment of Convex Programming Problems. A nonlinear convex programming problem is solved by methods of interval arithmetic which take into account the input errors and the round-off errors. The problem is reduced to the solution of a nonlinear parameter dependent system of equations. Moreover error estimations are developed for special problems with uniformly convex cost functions.

Intervallanalytische Behandlung konvexer Optimierungsaufgaben. Es wird ein nichtlineares konvexes Optimierungsproblem mit Hilfe der Intervallarithmetik gelöst, wobei die Eingangs- und Rundungsfehler berücksichtigt werden. Dieses Problem wird zurückgeführt auf die Lösung eines parameterabhängigen nichtlinearen Gleichungssystems. Außerdem werden Fehlerabschätzungen für spezielle Probleme mit stark konvexen Zielfunktionen angegeben.

1. Introduction

Till now linear programming problems are handled by methods of interval analysis by Machost [18], Krawczyk [15], and Beeck [5, 6]. Beeck gives a summary of the state-of-the-art in [6] and shows that the use of interval analysis for the treatment of linear programming problems has been successful. Nonlinear programming problems have been investigated by Dussel [9], Robinson [24], Mancini and McCormick [19], Oelschlägel and Süße [22, 23].

Data errors were considered in [22, 23] for the special case of quadratic programming problems. Therefore we want to consider a general convex programming problem with errors in data. We assume that the reader is well versed in interval analysis. A complete discussion can be found in Alefeld, Herzberger [1].

2. The Problem

Consider the nonlinear programming problem of the form

\[ \hat{z} (a, b_1, \ldots, b_l) = f (\hat{x} (a, b_1, \ldots, b_l); a) = \min_x f (x; a), \quad a \in [a] \in V_m (I (R)) \]

subject to

\[ g_i (x; b_i) \leq 0, \quad b_i \in [b_i] \in V_k (I (R)), \quad i = 1, \ldots, l \]

\[ x \geq 0, \]

(1)
where \( a, b_1, \ldots, b_l \) are vectors of parameters. The elements of \([a]\) and \([b_i]\), \(i=1, \ldots, l\) are tolerance intervals, in which the parameters vary independently. 

\([a], [b_1], \ldots, [b_l]\) are interval vectors.

All functions are assumed to have the following properties for any 

\[
a \in [a], b_i \in [b_i], i=1, \ldots, l.
\]

\[
f : \mathbb{R}^n \to \mathbb{R}^l, f \in C^1 (\mathbb{R}^n), f \text{ strictly convex in } \mathbb{R}^n,
\]

\[
g_i : \mathbb{R}^n \to \mathbb{R}^l, g_i \in C^1 (\mathbb{R}^n), g_i \text{ convex in } \mathbb{R}^n, i=1, \ldots, l.
\]

Furthermore all functions are to be continuously differentiable to all parameters. These derivations and \( f \) are continuous in \( \mathbb{R}^n \times [b_i], i=1, \ldots, n, \mathbb{R}^n \times [a] \).

We get the problem (1) from

\[
\hat{z} = f (\hat{x}) = \min f (x)
\]

subject to

\[
g_i (x) \leq 0, \ i=1, 2, \ldots, l
\]

\[
x \geq 0,
\]

where we have replaced data with errors by parameters.

To formulate the numerical problem, we introduce two sets.

**Definition 1:**

\[
\hat{\mathcal{X}} := \{ \hat{x} (a, b_1, \ldots, b_l)/a \in [a], b_i \in [b_i], i=1, \ldots, l \},
\]

\[
\hat{\mathcal{Z}} := \{ \hat{z} (a, b_1, \ldots, b_l)/a \in [a], b_i \in [b_i], i=1, \ldots, l \}.
\]

**Definition 2:** Denote the constraint set by

\[
\mathcal{P} (b_1, \ldots, b_l) := \{ x/g_i (x; b_i) \leq 0, x \geq 0, i=1, \ldots, l \}.
\]

If \( \hat{\mathcal{X}} \neq \emptyset \), then we want to find including sets for \( \hat{\mathcal{X}} \) and \( \hat{\mathcal{Z}} \) by the methods of interval analysis.

The question is now about the following interval vectors.

a) Compute an overhull \([x] \in V_n (I (R))\) of \( \hat{\mathcal{X}} \), i.e. \([x] \supseteq \hat{\mathcal{X}}\).

b) Compute the interval hull \([x]^H \in V_n (I (R))\), i.e. \([x]^H \supseteq \hat{\mathcal{X}}\) and \( d (X_i^H) = \min! \), \(i=1, \ldots, n\).

\[d ([a, \bar{a}]) = \bar{a} - a\] means the width of \([a, \bar{a}]\).

c) Compute an interior estimation \([x] \in V_n (I (R))\) of \( \hat{\mathcal{X}} \), i.e. \([x] \subseteq \hat{\mathcal{X}}\).

d) Compute a maximal interior estimation \([x] \in V_n (I (R))\), i.e. \([x] \subseteq \hat{\mathcal{X}}\) and \( \prod_{i=1}^n d (X_i) = \max! \).

e) Compute the same intervals for the unidimensional set \( \hat{\mathcal{Z}} \).

To compute some of these vectors we must know elementary properties of the sets \( \hat{\mathcal{X}} \) and \( \hat{\mathcal{Z}} \). The sets should not be neither compact nor connected. But we can guarantee under certain conditions these properties.