On the number of securities which constitute an efficient portfolio

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The purpose of this paper is to discuss the relationship between the number of securities which constitute an efficient portfolio as defined by the standard mean-variance portfolio selection model and the number of periods used to compute the efficient portfolio. It is shown that the number of data gives the upper bound of the number of securities which constitute an efficient portfolio, when each efficient portfolio is unique for a given expected return. Empirical tests based on actual return data show that this upper bound is very tight when the number of data is small. However, when more data are used, the upper bound becomes looser. This result is incompatible with the market efficiency. These empirical tests also indicate that a very tight upper bound often causes a degenerate case ensuring zero-variance portfolios.

\textbf{Keywords:} Standard mean-variance portfolio selection model, efficient portfolio, efficient frontier, market efficiency.

1. Introduction

A rational investor, who invests in a portfolio in the perfect market, is supposed to select a portfolio on an efficient frontier according to his own preference in risk and return. But, it has so far not been easy to plot an efficient frontier for a considerable number of securities, because large scale quadratic programming problems have to be solved repeatedly. Markowitz [5], the father of modern portfolio theory, proposed in the 1950's the Critical Line Method (CLM) to avoid the difficulty in computing efficient frontiers. However, it was not efficient enough to be used for practical purposes. A challenging work, making efficient frontier into an operational device, had been left to later researchers.

Recently, however, sophisticated algorithms have been developed to make it possible to plot an efficient frontier in a reasonable amount of time for a stock market in which more than a thousand securities are involved. Konno and Suzuki [4] proposed an algorithm using "a compact factorization of the covariance..."
Jensen and King [3] developed a decomposition method for the portfolio selection model, Takehara [12] applied an interior point method to portfolio selection and Markowitz et al. [7] improved the Critical Line Method. Also, Nakasato and Furukawa [8] developed a fast algorithm\(^1\) to compute an efficient frontier defined by the standard mean-variance portfolio selection model (the standard model). Using this algorithm, an efficient frontier can be calculated for all securities of the Tokyo Stock Exchange (TSE) within half a minute.\(^2\)

Empirical studies on the TSE market by Nakasato and Furukawa [9], using the newly developed algorithm, have revealed some unexpected results. One of them is that any efficient portfolio on the efficient frontier is composed of less than 60 securities. The other is that some degenerate cases are observed, when the covariance matrix is estimated from the return series of a few number of periods. Then, a zero-variance portfolio emerges on the efficient frontiers. These results are unexpected when using conventional portfolio selection models and equilibrium market theories such as the Capital Asset Pricing Model. In order to use the efficient frontier as an operational device, a variety of studies should be added and accumulated, as Nakasato and Furukawa's empirical studies have suggested. This paper is one of those studies.

In the next section, a portfolio selection model is defined. The active securities, that is, the securities constituting an efficient portfolio, are also defined. It will become clear that the number of active securities is limited to the number of periods used in estimating the covariance matrix. In solving an efficient frontier, a covariance matrix of returns of all securities is usually calculated from a series of historical returns of each security. So, the number of periods becomes one of the important factors. The third section shows the empirically observed relationship between the number of active securities and the number of periods used in estimating covariance matrices for the standard model. As a result, it becomes clear that, in the TSE market, some 60 securities compose an efficient portfolio when 120 periods are used for estimating the covariance matrix. The reasons for this will be discussed in this section. Finally, in the fourth section we conclude the paper with some suggestions for future research.

2. The standard model and a relationship between active securities and number of periods

The model referred to in this paper is the standard mean-variance portfolio selection model. In this model, investment decisions are subject to a budget

\(^1\)This algorithm is customized to solve the standard model using the parametric nonlinear programming method. Its concept is similar to the critical line method developed by Markowitz [5, 6].

\(^2\)Using our code based on this algorithm, it takes less than 30 seconds to compute a whole efficient frontier on a small workstation with 3 MFLOPS floating computation power in the case of a thousand securities investable and 120 sets of monthly return data used.