A LEVEL SET ALGORITHM FOR A CLASS OF REVERSE CONVEX PROGRAMS

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Abstract

A new algorithm is presented for minimizing a linear function subject to a set of linear inequalities and one additional reverse convex constraint. The algorithm utilizes a conical partition of the convex polytope in conjunction with its facets in order to remain on the level surface of the reverse convex constraint. The algorithm does not need to solve linear programs on a set of cones which converges to a line segment.

1. Introduction

A constraint $g(x) \leq 0$ is called a reverse convex constraint if $g$ is a quasi concave function. Optimization problems with reverse convex constraints generally induce nonconvex feasible regions which are often disconnected into several nonconvex parts. As a result, problems with such constraints generally have local optima which are not global optima.

Problems of this form were first studied by Rosen [15] in a control theoretic setting and subsequently, in an engineering setting by Avriel and Williams [1,2]. For a recent interesting application in VLSI design, see Vidigal and Director [24] and Thach [19]. Rosen developed a successive linearization technique which converges to a Kuhn–Tucker point. Meyer [9], in a more general setting, proves convergence to a Kuhn–Tucker point and it is from that paper that the term "reverse convex" is taken.

Ueing [23] was the first to consider global optimization for problems with reverse convex constraints. In fact, the problem Ueing considers is the minimization of a concave function subject to only reverse convex constraints and he develops a combinatorial approach based upon maximizing the objective subject to various combinations of reversals of the reverse convex constraints. Subsequently, Bansal and Jacobsen [4,3] studied the global optimization of a reverse convex program which represented the maximization of network flow capacity. In
particular, the incremental capacity cost functions were concave and, hence, represented economies-of-scale. Hillestad [8] then developed an edge search procedure for a linear program with one additional reverse convex constraint. Subsequently, Hillestad and Jacobsen [6] studied optimization problems with only reverse convex constraints and showed the convex hull of the feasible region is a convex polytope. They also suggested a cutting plane procedure, based upon Tuy cuts [20], as a possibly useful procedure for finding a good solution; however, they demonstrated that such a method need not converge to a feasible solution. Hillestad and Jacobsen [7] then presented an algorithm, for linear programs with one additional reverse convex constraint, which relies on simplex pivots and vertex enumeration of the feasible region intersected with the hyperplane determined by the current objective value. Tuy [22] then developed a method for convex programs with one additional reverse convex constraint which relies upon being able to solve concave minimization problems. In the latter paper, Tuy also shows that several reverse convex constraints can be converted to one such constraint at the expense of introducing an additional convex constraint and an additional variable. Tuy [21] also shows that virtually any optimization problem can, theoretically, be approximated by a convex program with one additional reverse convex constraint. Indeed, this latter result justifies the importance of this class of optimization problems.

In addition to the above relevance of reverse convex programs, insight is gained into the computational complexity of this class by observing that the 0–1 linear integer programming problem is equivalent to the associated bounded variable linear program with the additional reverse convex constraint \( \sum (x_i - x_i^2) \leq 0 \). Similarly, the minimization of a concave function \( f \) subject to linear constraints is obviously convertible to a linear program with one additional reverse convex constraint; in particular, choose an additional variable, \( x_{n+1} \), and minimize \( x_{n+1} \) subject to the linear constraints and the additional reverse convex constraint \( f(x) - x_{n+1} \leq 0 \).

Because of the close relationship between reverse convex programs and concave minimization, the reader is referred to the recent survey article by Pardalos and Rosen [14]. For additional papers on reverse convex programming which use, in one way or another, Tuy cuts [20] see, for example, [16,18,11,5].

2. Definitions

We briefly introduce the notation to be used throughout the paper. We denote by \( F_A \) a bounded convex polytope in \( \mathbb{R}^n \) defined by a system of linear inequalities; that is

\[
F_A = \{ x \in \mathbb{R}^n \mid Ax \geq b \},
\]