The Lagrange function is a source for getting the so-called "dual bounds" for a wide class of mathematical programming problems of the form: find

\[ f^* := \inf_{x \in X} f_0(x), \quad X \subseteq \mathbb{R}^n \] (1)

subject to the constraints

\[ f_i(x) \leq 0, \quad i = 1, \ldots, m. \] (2)

Let \( u \) be a vector of Lagrange multipliers, and let

\[ L(x, u) := f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \] (3)

be the Lagrange function.

Consider the problem

\[ \Psi(u) := \inf_{x \in X} L(x, u). \] (4)

For simplicity, let \( X \) be compact, and \( f_i, i = 0, \ldots, m \) be continuous functions. Then \( \Psi(u) \) is a concave function finitely determined for \( u \in \mathbb{R}^m \). For \( u \geq 0 \) and for any feasible point \( x \), we have \( L(x, u) \leq f_0(x) \) and \( \Psi(u) \) is a lower bound of \( f^* \). Let

\[ \Psi^* := \sup_{u \geq 0} \Psi(u). \] (5)

It is clear that \( \Psi^* \leq f^* \). The value \( \Psi^* \) is called the "dual bound" for problem (1)-(2).

Let \( u = \bar{u} \) and the minimum in (4) be attained on the set \( M(\bar{u}) \). Then the supergradient set \( G_{\Psi}(\bar{u}) \) of \( \Psi \) at the point \( \bar{u} \) is defined as follows [1]:

\[ G_{\Psi}(\bar{u}) = \text{co}\left\{ \{ f_i(x) \}_{i=1}^{m} : x \in M(\bar{u}) \right\}. \]

If \( M(\bar{u}) \) consists of a single point \( x(\bar{u}) \), then \( \Psi(u) \) is continuously differentiable at \( \bar{u} \) and its gradient is \( g_{\Psi}(\bar{u}) = \{ f_i(x(\bar{u})) \}_{i=1}^{m} \). In general, the determination of \( \Psi^* \) is a nondifferentiable optimization problem.

Problem (5) can be considered as a coordinating problem in the decomposition scheme with respect to constraints. We have constraints in two forms: (a) \( x \in X \), and (b) \( f_i(x) \leq 0, \quad i = 1, \ldots, m. \) Constraints of type (b) are accounted for in the
objective function with undefined Lagrange multipliers. We solve the “inner problem”, (4), for fixed \( \tilde{u} \) and find \( \Psi(u) \), some \( x(u) \in M(u) \), and the supergradient \( g_{\Psi}(u) = \{ f_i(x(u)) \}_{i=1}^m \). This information is sufficient for applying one of the nondifferentiable optimization methods for the solution of the coordinating problem (5) [2].

Let us consider a class of quadratic optimization problems of the following type: to find

\[
f^* := \inf_{x \in \mathbb{R}^n} K_0(x),
\]

subject to the constraints

\[
K_i(x) = 0, \quad i = 1, \ldots, m,
\]

where \( K_i(x) = (a_i x, x) + (b_i, x) + c_i, \ i = 0, 1, \ldots, m \). Let \( P^+ \) be a class of positive definite \( n \times n \) matrices and \( P^+ \) its closure. The Lagrange function for (6)-(7) equals

\[
L(x, u) = (A(u)x, x) + (b(u), x) + c(u),
\]

where \( A(u) = A_0 + \sum_i A_i u_i, \ b(u) = b_0 + \sum_i b_i u_i, \ c(U) = c_0 + \sum_i c_i u_i \). Let us set \( \Omega = \{ u \in \mathbb{R}^m: A(u) \in P^+ \} \) (accordingly \( \overline{\Omega} = \{ u \in \mathbb{R}^m: A(u) \in P^+ \} \)). If \( u \in \Omega \), then \( g'(u) = \min_x L(x, u) \) can be found by solving a linear system and we find:

\[
\Psi(u) = -\frac{1}{2} (A^{-1}(u) b(u), b(u)) + c(u).
\]

If in the formula \( \Psi^* = \max_{\Omega} \Psi(u) \) the maximum is attained on \( \Omega \), then \( \Psi^* = f^* \) and \( x(u^*) \) is an optimal solution of (6)-(7). Otherwise, the maximum (if any exists) is attained on the boundary: \( u^* \in \Omega \setminus \Omega \) and it is not defined uniquely. The function \( \Psi(u) \) is differentiable on \( \Omega \). Note that for almost all points \( \tilde{u} \in \overline{\Omega} \setminus \Omega \), we have \( \lim_{u \in \Omega, u \to \tilde{u}} \Psi(u) = -\infty \). This means that \( \Psi(u) \) is similar, by its properties, to barrier functions and allows one to use an unconstrained optimization technique for the evaluation of \( \Psi^* \).

Dual quadratic bounds can be used for getting the global extremum in polynomial programming problems [4,5], which consist of finding:

\[
\min P_0(z)
\]

subject to

\[
P_i(z) = 0, \quad i = 1, \ldots, m,
\]

where \( P_i(z), i = 0, \ldots, m \) are polynomials of \( z_1, \ldots, z_n \). By introducing new variables and making use of quadratic substitutions of the form \( z_i^2 = y_i, \ z_{jk} = z_j z_k \), and so forth, we can reduce the polynomials in (8) to quadratic functions of the extended variable set. Hence, any problem of type (8) can be reduced to a quadratic optimization problem. We can apply the dual-bounds approach to solve this problem. It would be interesting to find classes of polynomial problems for which the quadratic dual bound \( \Psi^* = f^* \). This question was investigated for a problem of finding an unconstrained global minimum of a polynomial function.