A NOTE ON THE 'ONE-LINK' INTEGRAL OF U(N) LATTICE GAUGE THEORY

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ABSTRACT. We calculate the 'one-link' U(N) integral in closed form by a direct method, i.e., polar decomposition and integration over angular variables. The result agrees with the known solution of the Brower–Nauenberg equation, at least for \( N \leq 4 \).

1. INTRODUCTION

The integral over the U(N) group

\[
Z(A) = \int_{U(N)} [dU] \exp \{ \text{Tr}(AU^+U^*A^*) \}
\]

where \([dU]\) is the Haar measure on U(N) and \( A \) is any complex matrix, has recently been the object of intensive investigations [1] – [3]. Its interest has been stressed by many authors, so that we shall limit ourselves here to recall that we may consider \( A = g_0^2 \langle U \rangle \) as an average self-consistent plaquette; \( Z(A) \) is an essential ingredient of a mean-field type approximation of lattice gauge theory.

A closed form of \( Z(A) \) has been proposed by Brower, Rossi and Tan [4] who showed that

\[
\int \frac{d\lambda}{2\pi i} \frac{\det \lambda F^{-1} - (2\lambda)}{\det \lambda F^{-1}(2\lambda)} = N \det \lambda F^{-1}(2\lambda) \]

is a solution of the Brower–Nauenberg equation \((\lambda_1, ..., \lambda_N)\) being the eigenvalues of \( \sqrt{AA^+} \). It is not clear, however, whether the only boundary condition \( Z(0) = 1 \) is sufficient to give a unique solution. It is true that the asymptotic growth at infinity \((\lambda \to \infty, \text{ or equivalently } g_0 \to 0)\) is correct, as it can be obtained by a Gaussian approximation directly from Equation (1). However, no rigorous proof of uniqueness under these conditions was given, to our knowledge. On the other hand, the system of partial differential equations introduced in Reference [3] leads to a unique solution, independently of the growth at infinity, but it is difficult to check whether, \( Z(\lambda) \) is, in fact, a solution (it was a painful experience to check it for \( N = 3 \) – see also Section 3).

A more direct derivation of Equation (2) then seems desirable, and this note is devoted to this aim.**

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** Notice that in Reference [7] a direct integration was given for \( N = 2 \) and \( N = 3 \). However, the identification with Equation (2) was not given explicitly, but on the basis on the assumed uniqueness of the solution.
The result depends on an integral formula of Itzykson and Zuber [5] and Mehta [6] which allows
the integration over angular variables in the polar decomposition. The final form of Equation (2)
is obtained by applying a new identity satisfied by Bessel functions.

It is hoped that this direct method will also apply to SU(N), with minor modifications.

2. THE INTEGRAL OVER U(N) USING THE POLAR DECOMPOSITION

We let $\Lambda = \Lambda$ be diagonal with non-negative entries $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let us introduce polar co-
ordinates in U(N):

$$U = S \Phi S^+$$

where $\Phi = \text{diag} \left[ e^{i\varphi_1}, \ldots, e^{i\varphi_N} \right]$ and $S$ runs over some coset space. It is well known that the integral
Equation (1) becomes

$$Z(\lambda) = \frac{2^N(N-1)}{N!(2\pi)^N} \int_0^{2\pi} \ldots \int_0^{2\pi} [d\varphi_1 \ldots d\varphi_N] \prod_{i>j} \left( \sin \frac{\varphi_i - \varphi_j}{2} \right)^2 \exp \left\{ 2 \text{ Tr} (\Lambda S (\Phi S^+)) \right\}$$

where $C = \text{diag} \left[ \cos \varphi_1, \ldots, \cos \varphi_N \right]$. A result of Reference [5] allows us to integrate over $S$ to
obtain

$$Z(\lambda) = \frac{2^{(1/2)N(N-1)} \Pi^{N-1} \nu!}{N!(2\pi)^N} \int_0^{2\pi} \ldots \int_0^{2\pi} [d\varphi_1 \ldots d\varphi_N] \prod_{i>j} \left[ \frac{\sin^2 (\varphi_i - \varphi_j/2)}{(\lambda_i - \lambda_j)(\cos \varphi_i - \cos \varphi_j)} \right] \det \left\{ e^{2\lambda_j \cos \varphi_k} \right\}$$

It is convenient to introduce complex variables $z_k = e^{i\varphi_k}$, in terms of which we find

$$Z(\lambda) = \frac{\Pi^{N-1} \nu!}{N! \Delta(\lambda)} \oint \ldots \oint [dz_1 \ldots dz_N] \frac{\Delta(z)}{\Pi_{i>j}(1-z_i z_j)} \det \left\{ e^{\lambda_j (z_k + z_k^{-1})} \right\}$$

where $\Delta(\cdot)$ denotes the Vandermonde determinant. We can now simplify this expression by
writing the determinants explicitly:

$$\Delta(z) \det \left\{ e^{\lambda_j (z_k + z_k^{-1})} \right\}$$

$$= \sum_p \delta_p \prod_{j=1}^N z_j^{l_j} \sum_{Q} \delta_Q \prod_{k=1}^N e^{\lambda_k (z_Q + z_Q^{-1})}$$

$$= \sum_p \sum_{R} \delta_p \delta_{R^{-1}} \prod_{j=1}^N z_j^{l_j} \prod_{k=1}^N e^{\lambda_k (z_P + z_P^{-1})}$$

where $P, Q, R$ run over the permutation group, $Q = PR^{-1}$ and $\delta_p$ is the character of $P$ in the