THE MATHEMATICAL THEORY OF FACTORIAL INVARIANCE UNDER SELECTION

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It is first demonstrated that Aitken's selection formulas are equivalent to a linear transformation in the factor space. On this basis the Thomson-Ledermann theorem concerning the invariance of the number of common factors under selection, and a theorem concerning the invariance of factor loadings under selection are derived. A mathematical proof of the results of Thurstone, which are concerned with the invariance of simple structure under selection, is given. The paper provides a conclusive answer to the question, considered by Thurstone and Thomson, whether a multivariate selection is always reducible to successive univariate selections.

I. Selection as a Linear Transformation

The effect of a selection of population (i.e., selection of subjects in psychological experiments) on correlations and variances is indicated by Aitken's formulas

\[ V_{ik} = V_{ii}R_{ik}^{-1}R_{ki} = V_{ki} \]

\[ V_{kk} = R_{kk} - R_{ki}(R_{ii}^{-1} - R_{ii}^{-1}V_{ii}R_{ii}^{-1})R_{ik} \]

(see 1, 2, 3 or 4), involving the inverse \( R_{ii}^{-1} \). As is well known, the inverse of a square matrix exists only if the matrix is nonsingular, i.e., if its determinant does not vanish. In virtue of this, we must have \( |R_{ii}| \neq 0 \). On the other hand, it follows from this condition that the rows and the columns of \( R_{ij} \) must be linearly independent. Accordingly, we can state the following theorem:

**Selection Theorem I:** Selection tests are linearly independent in a selection determined by Aitken's formulas.

We then pass over to a geometrical mode of representation customary in factor analysis and let each test be represented by a vector of unit length. We call the manifold spanned by the \( l \) "selection test vectors" (i.e., vectors that geometrically represent the selection tests) the "selection space." Since, in virtue of Selection Theorem I, the \( l \) selection test vectors are linearly independent, the selection space is \( l \)-dimensional. It follows directly from the definition of the selection space that selection test vectors are entirely in this space. Other test vectors, on the other hand, may have—in addition to a selection space component—another component outside of this space. We then set up an arbitrary coordinate system in the selection space and express
the projections of the selection test vectors on the coordinate axes by the matrix \( ||a_{jm}|| = A_j \), where the representative element \( a_{jm} \) expresses the projection of the \( j \)th selection test vector on the \( m \)th coordinate axis. On the other hand, let the matrix \( ||a_{km}|| = A_k \) express the projections of the non-selection test vectors on the coordinate axes of the selection space, the representative element \( a_{km} \) expressing the projection of the \( k \)th non-selection test vector on the \( m \)th coordinate axis. The scalar products of selection test vectors and the selection space components of non-selection test vectors then have the form

\[
\rho_{ij} = \sum_m a_{jm}a_{jm},
\]

\[
\rho_{ik} = \sum_m a_{jm}a_{km},
\]

\[
\rho_{kK} = \sum_m a_{km}a_{Km},
\]

where \( j \) and \( J \) refer to selection tests and \( k \) and \( K \) to non-selection tests. As the selection test vectors are entirely in the selection space, their inter-correlations are simply

\[
\tau_{ij} = \rho_{ij},
\]

\[
\tau_{ik} = \rho_{ik}.
\]

On the other hand, as the non-selection test vectors are not necessarily entirely in the selection space, we have

\[
\tau_{kK} = \rho_{kK} + \bar{\rho}_{kK},
\]

where \( \bar{\rho}_{kK} \) is that part of the scalar product of the test vectors \( k \) and \( K \) which originates outside of the selection space. In matrix notation, we can express the foregoing as

\[
R_{ij} = A_jA_j',
\]

\[
R_{ik} = A_jA_k',
\]

\[
R_{kk} = A_kA_k' + \bar{R}_{kk}.
\]

In the selection space we then perform a linear transformation, converting the selection test vectors \( j \) into new vectors whose projections on the coordinate axes are \( b_{jp} \). Transformation coefficients \( l_{mp} \), \( l' \) in number, are determined by \( l^2 \) linear equations,

\[
b_{jp} = \sum_m a_{jm}l_{mp}. \quad (j = 1, 2, \cdots , l)
\]

\[
(p = 1, 2, \cdots , l)
\]

The new coordinates of the non-selection test vectors can then be determined from equations