CONNECTIVITY AND GENERALIZED CLIQUES IN SOCIO METRIC GROUP STRUCTURE

R. DUNCAN LUCE
GRADUATE STUDENT, DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

By using the concepts of antimetry and \( n \)-chain it is possible to define and to investigate some properties of connectivity in a sociometric group. It is shown that the number of elements in a group, the number of antimetries, and the degree of connectivity must satisfy certain inequalities. Using the ideas of connectivity, a generalized concept of clique, called an \( n \)-clique, is introduced. \( n \)-cliques are shown to have a very close relationship to the existence of cliques in an artificial structure defined on the same set of elements, thus permitting the determination of \( n \)-cliques by means of the same simple matrix procedures used to obtain the clique structures. The presence of two or more \( m \)-cliques, where \( m \) is the number of elements in the group, is proved to mean an almost complete splitting of the group.

1. Introduction

This paper is devoted to extending the theoretical and practical mathematical results presented in an earlier paper (3). In that paper it was shown that certain elementary matrix operations permit, to some extent, an analysis of the simple type of psychological group structure which is often expressed by a sociometric diagram. Specifically, we envisage a finite set of \( m \) (\( \geq 2 \)) elements \( i, j, k, \ldots \) having a structure defined on them as follows: For any two elements \( i \) and \( j \) of the set there either exists or does not exist some one type of "directional" relationship from \( i \) to \( j \). This one type of relationship on the group may assume various forms such as communication or friendship; and so to acquire generality in the discussion we shall speak of an antimetry from \( i \) to \( j \). Then communication is merely a special type of antimetry. It is arbitrarily assumed that no antimetry can exist from an element \( i \) to itself. By "directional" is meant that knowing of the existence or non-existence of an antimetry from \( i \) to \( j \) does not give any information about the existence or non-existence of an antimetry from \( j \) to \( i \). For example, if the form of the antimetry is "\( i \) communicates to \( j \)," then the specific knowledge that \( a \) can communicate to \( b \) does not tell us, in general, whether \( b \) can communicate to \( a \).
To such a structure of antimetries we associate a matrix $G = [g_{ij}]$ as follows: If there exists an antimetry from $i$ to $j$ then write the number 1 in the $g_{ij}$ entry of the matrix, and if no such antimetry exists place a 0 in the entry. Since we assume no antimetry can exist from $i$ to $i$, the main diagonal will always be 0's. If there is both an antimetry from $i$ to $j$ and from $j$ to $i$, then we say a symmetry exists between $i$ and $j$. From the matrix $G$ we may extract a symmetric matrix $S$, called the matrix of symmetries, as follows: If a symmetry exists between $i$ and $j$ then $s_{ij} = s_{ji} = 1$ and otherwise $s_{ij} = s_{ji} = 0$.

In (3) there are two theoretical results which will be utilized here and which, for convenience, we shall briefly summarize. However, where the method of determining cliques is necessary to apply the results of Sections 2, 3, and 6, it will be assumed that the reader is familiar with the procedure as outlined in that paper.

Let us define an $n$-chain from $i$ to $j$ as an ordered set of $n+1$ elements, with $i$ the first and $j$ the last, such that there is an antimetry from $i$ to the second element, an antimetry from the second to the third, etc. and an antimetry from the $n$th element to $j$. If the antimetry is communication then an $n$-chain is a path of indirect communication from $i$ to $j$ in $n$ steps. Two $n$-chains from $i$ to $j$ will be considered to be distinct if the $k$th element of the first chain differs from the $k$th element of the second for some value of $k$ between 2 and $n$. It was shown that a positive integer $c$ occurs in the $ij$ entry of the $n$th power of $G$, i.e., $g_{ij}(n) = c$, if and only if there are $c$ distinct $n$-chains from $i$ to $j$.

An interpretation of this result in terms of the communication of information from element to element may give some insight into this formalism. We can imagine that the structure matrix $G$ represents a linear transformation of a vector of information in the following way: Let $a_i$ be the symbolic representation of the information introduced into the group at element $i$, then the row vector $A = [a_1, a_2, \ldots, a_m]$ represents all the information introduced into the group. If we suppose that each element transmits all the information it has to each other element to which it is connected by an antimetry, then element $j$ will receive the information $\sum_{i=1}^{m} a_i g_{ij}$. In order to permit a matrix interpretation of this sum, addition will have to follow the law $k + k = 2k$, in which case the operation indicated is the matrix product $AG$, which is a row vector representing the distribution of information in the group after one inter-