A GENERALIZATION OF THURSTONE'S LEARNING FUNCTION*

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Thurstone's equation giving the probability of a correct response \( p \) as a function of practice time \( t \) when punishment and reward have equal effects has been generalized to the case where the effect of punishment is not necessarily equal to the effect of reward. Since the general equation is somewhat unwieldy, three special cases are considered, where reward has no effect, where punishment has no effect, and where these effects are equal. Equations are given together with tables for making a rectified plot for each of the three special cases.

This paper presents a generalization of the learning function developed by Thurstone (2) and will discuss certain interesting special cases of this general equation.

Definitions and Assumptions

Following Thurstone's development we will define the following variables:

\[ s = \text{the strength of the correct response, or the number of unit correct responses available to the organism at any moment.} \]

\[ e = \text{the strength of the incorrect response, or the number of unit incorrect responses available to the organism at any moment.} \]

\[ p = \text{the probability of a correct response.} \]

\[ q = \text{the probability of an incorrect response.} \]

\[ t = \text{practice time.} \]

The relationship between \( p, q, s, e, \) and \( t \) is assumed to be given by the following equations:

\[ p = \frac{s}{s + e} \quad (1) \]

and

\[ q = \frac{e}{s + e} \quad (2) \]

*This study was supported in part by contract N6onr 270-20 between the Office of Naval Research and Princeton University. The opinions expressed are, of course, those of the author and do not represent attitudes or policies of the Office of Naval Research.
These two assumptions are identical with assumptions [1] and [2] from Thurstone (2). It is also assumed that the variation of $s$ and $e$ with respect to time is given by

$$\frac{ds}{dt} = kp, \tag{3}$$

and

$$\frac{de}{dt} = -cq. \tag{4}$$

These two assumptions are similar to Thurstone's assumptions [4] and [6] but are more general because it is not assumed that $c = k$. In Thurstone's development it was assumed that the effect of rewarding the correct response ($k$) was equal to the effect of punishing the incorrect response ($c$). Here the more general case in which these parameters may be either the same or different is being considered.

From the foregoing set of four equations it is possible to derive the functional relationship between the probability of a correct response ($p$) and the practice time ($t$). It might be noted that the assumptions used here are essentially the same as those used in a former paper (1) except that there the functional relationship between cumulative errors and cumulative correct responses was obtained, while here the interest is in the relationship between two other variables, practice time ($t$) and the probability of a correct response ($p$).

**Derivation of the General Case**

Substituting (2) in (4) gives

$$\frac{de}{dt} = -\frac{ce}{s + e}. \tag{5}$$

If $s$ is expressed as a function of $e$, and this function substituted in (5), (5) will then be a differential equation in $e$ and $t$ only and may be integrated readily. We can obtain $s$ as a function of $e$ by the following procedure: Dividing (3) by (4) gives

$$\frac{ds}{de} = -\frac{k}{c} \frac{p}{q}. \tag{6}$$

Dividing (1) by (2) gives

$$p = \frac{s}{q - e}. \tag{7}$$

Substituting (7) in (6) we have

$$\frac{ds}{de} = -\frac{k}{c} \frac{s}{e}. \tag{8}$$