ORTHOGONAL INTER-BATTERY FACTOR ANALYSIS*

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It is the purpose of this paper to present a method of analysis for obtaining (i) inter-battery factors and (ii) battery specific factors for two sets of tests when the complete correlation matrix including communalities is given. In particular, the procedure amounts to constructing an orthogonal transformation such that its application to an orthogonal factor solution of the combined sets of tests results in a factor matrix of a certain desired form. The factors isolated are orthogonal but may be subjected to any suitable final rotation, provided the above classification of factors into (i) and (ii) is preserved. The general coordinate-free solution of the problem is obtained with the help of methods pertaining to the theory of linear spaces. The actual numerical analysis determined by the coordinate-free solution turns out to be a generalization of the formalism of canonical correlation analysis for two sets of variables. A numerical example is provided.

I. Introduction

Inter-battery factor analysis as devised by Tucker [1958] proposes to investigate factors which account for the intercorrelations between two sets of tests. While Tucker’s purpose was to check on the generalization of hypothesized factors across batteries of measures constructed to involve the hypothesized factors, an alternative purpose is to isolate from the set of factors required to describe the intercorrelations of the combined battery, those factors common to the two sub-batteries.

Consider a battery of tests made up of two sub-batteries, 1 and 2. The total correlation matrix $R$ can then be partitioned into four parts:

\[
\begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
\]

$R_{11}$ contains only the correlations of tests of battery 1, $R_{22}$ contains the correlations within battery 2, and $R_{12} = R_{21}$ is the matrix of correlations between tests of the two batteries. If $F^*$ is a factor matrix producing all correlations in $R$, $R = F^*F^{**}$, then $F^*$ may be considered a supermatrix composed of several submatrices:

\[
F^* = \begin{bmatrix}
U & G & 0 \\
V & 0 & H
\end{bmatrix}
\]

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$U$ and $V$ are matrices of factors which appear in both batteries, the "inter-battery factors," accounting for the intercorrelations in $R_{12}$. Only battery 1 has nonzero loadings in factors of $G$; similarly the factors of $H$ are present in battery 2 only. These factors may be conveniently labeled "battery specific."

It is obvious that the matrix equation

$$R_{12} = UV'$$

holds. Applying the Eckart-Young [1936] procedure, Tucker gave solutions for (3) in terms of $U$ and $V$ when $R_{12}$ is known and thus determined sets of inter-battery factors. It is clear that inter-battery factor analysis would be an important tool in psychological research.

Of course, the solution of (3) is not unique. However, the rotation problem is much more difficult than in ordinary factor analysis, as Tucker pointed out, even if $U$ and $V$ are required to be of minimal rank. For, if $M$ is any nonsingular square matrix of appropriate order, the matrices $U^* = UM$ and $V^* = V(M')^{-1}$ constitute a solution of (3) provided that $U$ and $V$ are a solution. Since $M$ may be chosen almost arbitrarily, serious violations of basic requirements may result. For example, we may get negative residual communalities or residual correlations exceeding unity in $R$ after the extraction of inter-battery factors. These pitfalls are avoidable in principle by carefully choosing $U$ and $V$ [see Gibson, 1960; 1961a], but the mere possibility of their occurrence leaves one uneasy.

Furthermore, it would seem natural to require that a finally chosen set of inter-battery factors should not prejudice the interpretability of the battery specific factors even if the latter are not of interest to the investigator. And finally, it is impossible to make any statements as to the orthogonality or obliqueness of the inter-battery factors; no information is given concerning the angles.

Under these circumstances doubts as to the reliability of inter-battery factor analysis are inevitable. It seems that only few empirical studies involving this type of analysis have been undertaken so far, e.g., by Rulon, Schweiker and Garfunkel [1965] and Beaton [1966].

Gibson's [1961b] asymmetric approach to multiple-factor analysis has been introduced as an alternative to Tucker's inter-battery factor analysis. Gibson's procedure is devised to yield orthogonal inter-battery factors if some underlying hardly verifiable assumptions are met. The investigator must be able to specify two groups of tests which span the same space if the method is to work. Apart from Gibson's own illustration, no study employing this method has been published so far. Basically the same approach has been used recently by Gibson [1963] for some related purposes in factor analysis.

Both methods, Tucker's as well as Gibson's, claim the advantage of