A COMPARISON OF FOUR METHODS OF CONSTRUCTING FACTOR SCORES*

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Four least-squares methods for constructing factor scores have been described in the literature. The formal properties of these scores are developed, and they are compared in terms of four generally desirable properties of constructed factor scores. In particular, it is shown that two of the methods yield scores that are conditionally unbiased, and univocal in the sense of Guilford and Michael, though not orthogonal, while one of the other methods yields orthogonal scores.

It is shown that constructed factor scores cannot be simultaneously univocal and orthogonal, unless we choose the special basis in factor space given by Canonical Factor Analysis.

The general problem of choosing between the methods is discussed, on the basis of the theoretical relations obtained.

1. Introduction

In the past thirty years or so the theory of common factor analysis has undergone a tremendous amount of development. Almost all of this development has been concerned with methods for the determination of matrices of factor loadings, and methods for transforming these into an "acceptable" form for interpretation. Very little attention has been given to the correlative problem of determining factor scores.

From one point of view, this is not surprising, since the usual objectives of factor analysis are commonly considered to be attained when one has an interpretable factor structure. On the other hand, the theoretical problems connected with the indeterminacy of factor scores [Guttman, 1955] have led to misgivings about the usefulness of the common factor model, and some investigators [Guilford and Michael, 1948; Heermann, 1963] have been concerned to seek methods for constructing factor scores that may be used in a further phase of research, following on from the initial factor analysis itself. Further, Horst [1965] and Burkett [1964] show that reduced-rank approximations to a set of multivariate observations have useful properties in multiple regression problems. Also, the method of nonlinear factor analysis described by McDonald [1962, 1967] requires, loosely speaking, knowledge

*EDITOR'S NOTE: The reader will quickly discover that this article develops several of the generalizations given in the second part of the preceding article, "On Factors and Factor Scores." Independent development of the same generalizations is, of course, not a new phenomenon. Because the Presidential Address automatically is accepted for publication and given space in the December issue, it was decided that the only fair thing to do was to print this article in the same issue.
about factor scores, and it is not immediately evident whether the methods recommended for obtaining such knowledge are the most suitable for this purpose. The primary motive for the present paper was to examine this last question.

It has commonly been pointed out that factor scores cannot be determined precisely, but only "estimated", since the number of common and unique factor scores exceeds the number of observed variables. Two defects of the most commonly used "estimates" (due to Thurstone, 1935) are, firstly, that even when the "true" factor scores in the model are assumed mutually orthogonal, the "estimates" are typically correlated, and secondly, the estimated scores do not correlate only with the corresponding true factor scores, but they also have non-zero correlations with non-corresponding factors in the model.

Given a $m \times r$ matrix of factor loadings for $m$ fixed tests, $m > r$, from an observed $m \times 1$ vector it is possible to construct a $r \times 1$ vector, such that this will be, in some sense, an approximation to the corresponding $r \times 1$ vector of factor scores in the model. For convenience we will refer to the components of the former as constructed factor scores, and the components of the latter as true factor scores. For the purposes of this paper it will suffice to consider orthogonal models only.

In the context of the orthogonal factor model, there are at least four properties of a vector of constructed factor scores that may be generally desirable. Let $x^\circ = [x_1^\circ, \ldots, x_r^\circ]$, be a vector of true factor scores, such that $\Sigma \{x_0x_0^*\}$ is a diagonal matrix, and $x' = [x_1, \ldots, x_r]$, be a corresponding vector of constructed factor scores. Then, firstly, $x$ should in some sense approximate $x_0$ as closely as possible. Secondly, the constructed factor scores should also be orthogonal, i.e. $\Sigma \{xx'\}$ should be a diagonal matrix. Thirdly, each constructed factor score should correlate zero with each non-corresponding true factor score, i.e. $\Sigma \{x_0x'\}$ should be a diagonal matrix. Constructed factor scores having this property have been called univocal by Guilford and Michael [1948]. Lastly, we might desire the property that the constructed factor scores be conditionally unbiased estimators of the corresponding true factor scores, that is $\Sigma \{x \mid x_0\} = x_0$ (the conditional expectation being taken over the subpopulation of "persons" whose true factor scores are $x_0$).

By far the most widely known method for obtaining constructed factor scores is the least squares procedure due to Thurstone. It is well known that, in general, factor scores so constructed do not possess the last three of the four properties listed above. (It should be noted that Heermann [1963] has described procedures whereby one may transform a set of constructed factor scores in the sense of Thurstone, into either a set having the second property or a set having the third property.)

In addition to the Thurstone method, three other least squares procedures