The solution of the problem of enumeration of the $n$-paths in a digraph has so far been attempted through an indirect approach of enumerating the redundant chains. The approach has yielded an algorithm for determination of the general formula for the matrix of redundant $n$-chains and also a partial recurrence formula for the same. This paper presents a direct approach to the problem. It gives a recurrence relation expressing the matrix of $n$-paths of a digraph in terms of the matrices of $(n-1)$-paths of its first-order subgraphs. The result is exploited to give an algorithm for computing the matrix of $n$-paths. The algorithm is illustrated with a $6 \times 6$ matrix.

1. The Problem

Group structure defined by a relationship between its members has been the subject matter of many investigations in sociometric studies. (A list of these is available in Harary’s review article [2].) A model consisting of a digraph and an adjacency matrix has been extensively used in these studies. The points in a digraph represent the members of the group and the directed lines, the relation between the members. The $ij$th entry in the adjacency matrix is put equal to one if there is a directed line from the point $i$ to the point $j$ in the digraph and zero, otherwise.

If a sequence of points $i = \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1} = j$ is such that there is a directed line from $\gamma_1$ to $\gamma_{l+1}$, for $l = 1, 2, \ldots, n$, then the sequence of points together with these directed lines is called a chain of steplength $n$ (or briefly an $n$-chain) from $i$ to $j$. If any two of these points are the same we call them a redundant pair. For example, if $\gamma_1 = \gamma_m$ for $l \neq m$ then $(\gamma_1, \gamma_m)$ is a redundant pair. An $n$-chain with at least one redundant pair is a redundant $n$-chain. An $n$-chain with no redundant pair is a nonredundant $n$-chain, or simply an $n$-path.

The problem of determination of the number of $n$-paths from any point $i$ to any other point $j$ of a digraph was first encountered by Festinger [1] in connection with a study of the spread of rumor in a social group. The num-
ber of \( n \)-paths may be derived either directly or by subtracting the number of redundant \( n \)-chains from the number of all \( n \)-chains, which is given by the \( ij \)th entry of the matrix \( X^n \), where \( X \) is the adjacency matrix of the digraph \( D \) of the group. As we shall see later, previous attempts at solving the problem were based on the indirect approach. This paper deals with an algorithm for the direct enumeration of the paths.

2. Notation and Terminology

The matrix giving the number of redundant \( n \)-chains from any point of the digraph \( D \) to any other point will be denoted by \( R_n(D) \), and the matrix giving the number of \( n \)-paths of \( D \) will be denoted by \( P_n(D) \).

The graph obtained by removing all the lines from and to a point \( j \) of \( D \) will be denoted by \( (D - j) \) and will be called a first-order subgraph of \( D \). Its adjacency matrix is obviously obtained by equating to zero the \( j \)th row and column of the adjacency matrix \( X \) of \( D \). It will be denoted by \( X_j \) and called a first derived matrix of \( X \). Thus, corresponding to the \( p \) points of \( D \) there are \( p \) first-order subgraphs of \( D \) and associated with them are the \( p \) first derived matrices of \( X \).

The graph obtained by removing all the lines from and to each of the \( r \) points \( p_1, p_2, \ldots, p_r \) of \( D \) will be denoted by \( (D - p_1 - p_2 - \cdots - p_r) \) and called an \( r \)-order subgraph of \( D \). Its adjacency matrix obtained by equating to zero the rows and columns of \( X \) numbered \( p_1, p_2, \ldots, p_r \) will be denoted by \( X_{p_1, p_2, \ldots, p_r} \) and will be called an \( r \)th derived matrix of \( X \). Thus there are \( C_r^p \) distinct \( r \)-order subgraphs of \( D \) and an equal number of distinct \( r \)th derived matrices of \( X \). The set of all distinct \( r \)th derived matrices of \( X \) shall be denoted by \( S_r \).

The matrix of redundant \( n \)-chains and the matrix of \( n \)-paths of the \( r \)-order subgraph \( (D - p_1 - p_2 - \cdots - p_r) \) will be denoted respectively by \( R_n(D - p_1 - p_2 - \cdots - p_r) \) and \( P_n(D - p_1 - p_2 - \cdots - p_r) \) or by \( R_n(X_{p_1, p_2, \ldots, p_r}) \) and \( P_n(X_{p_1, p_2, \ldots, p_r}) \).

3. Review of Previous Work

Luce and Perry [3] classified the redundant \( n \)-chains into five mutually exclusive and exhaustive categories \( A, B, C, D, \) and \( E \). Denoting the matrices of redundant \( n \)-chains in these categories by \( A_n, B_n, C_n, D_n, \) and \( E_n \), so that \( R_n = A_n + B_n + C_n + D_n + E_n \), they obtained a recurrence formula for \( A_n \), namely \( A_n = XR_{n-2}X \). No recurrence formulas for the other components of \( R_n \) could, however, be obtained. Their final result is a partial recurrence formula for \( R_n \) in terms of \( R_{n-1}, R_{n-2}, \) and \( D_n \). As this still involves \( D_n \) for which no recurrence formula or method of direct enumeration has been obtained, the formula is not readily usable.

Using a method of partitions to calculate \( R_n \), Ross and Harary [4] obtained explicit formulas for \( R_1 \) to \( R_6 \) and gave an algorithm for deriv-