NONLINEAR FACTORS IN TWO DIMENSIONS

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Intercorrelations among tests nonlinearly related to underlying dimensions require more linear factors than content would demand. For the case of two independent underlying content dimensions, a fictitious example is constructed and made to yield a transformation useful for the nonlinear analysis of certain empirical data. That transformation, when applied to a standard factorization (centroid or principal components if certain symmetries obtain) of the appropriate empirical correlations, yields parameters descriptive of plausible nonplanar regression surfaces for tests on the two underlying dimensions. An empirical example is presented and discussed.

The dilemma of difficulty factors has beset factor analysts for many years [cf. 1, 2, 3, 8, and 12]. When a group of tests quite homogeneous as to content but varying widely in difficulty is subjected to factor analysis, the result is that more factors than content would demand are required to reduce the residuals to a random pattern. This effect is generally attributed to curvilinear relations among the tests, such curvilinearties being forced by the differential difficulty of the tests. Coefficients of linear correlation, when applied to such data, will naturally underestimate the degree of nonlinear functional relation that exists between such tests. Implicitly, then, it is nonlinear relations among tests that lead to difficulty factors. Explicitly, however, the factor model rules out, in its fundamental linear postulate ([11], p. 68), only such curvilinear relations as may exist between tests and factors.

Some recent developments [4, 5] in the theory of latent profile analysis (the extension of Lazarsfeld’s latent structure analysis [10] to the study of interrelations among quantitative variables) have shed new light on the problem of difficulty factors. For the case of a single underlying dimension, there now exist nonlinear solutions which account for the intercorrelations as they stand. No attempt is made to obliterate the tell-tale difficulty pattern in the correlations by the choice of coefficient or by “corrections” or “adjustments” of any kind. The solutions account for the intercorrelations with precisely the same discrepancies as does the centroid factorization, for they are linear transformations of that factor matrix. They describe nonlinear regressions of tests on the single underlying dimension in a way that conforms

*The opinions expressed are those of the author and are not to be construed as reflecting official Department of the Army policy.
with commonsense expectations for easy and difficult tests. Regressions of easy tests on the underlying dimension are concave downward. The reverse obtains for difficult tests.

The purpose of the present paper is to provide a similar kind of analysis for a set of tests that are nonlinearly related to two statistically independent underlying dimensions. This will be done by developing an idealized fictitious example to a point where it provides a transformation that can be used empirically. The application of that transformation to a standard factorization (the centroid or principal axes if certain symmetries obtain) of an appropriate set of empirical correlations results in the desired nonlinear solution. Such an application will be illustrated on empirical data provided by Dingman [2].

![Figure 1](attachment:image)

**Figure 1**

*Theoretical Partitioning of a Normal Bivariate Surface for Uncorrelated X and Y*

Consider, then, the correlation surface represented by Figure 1. X and Y are two hypothetical statistically independent underlying dimensions. The correlation surface is normal bivariate, and the dotted lines divide it into nine sectors in such a way that every vertical and horizontal array has frequencies in the ratio 1:2:1. In other words, the standard score equations for the vertical and horizontal dotted lines are, respectively, $X = \pm .6745$ and $Y = \pm .6745$. The dashed lines divide the space into five regions, one