A COVARIANCE ANALYSIS OF MULTIPLE PAIRED COMPARISONS*

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In the method of multiple paired comparisons the dominance of object \( j \) over object \( k \) is observed upon \( p \) attributes. The present paper develops a covariance analysis for these paired comparisons in terms of a linear model which includes scale, bias, and interaction effects, along with \( s \) covariants upon which the comparisons are presumably dependent. The covariance model gives rise to adjusted parameter estimates and hypothesis tests for the residual pairwise layout from which the effects of the \( s \) covariants have been removed. These estimation and testing procedures are illustrated with an analysis of political judgment data, and their relevance to the general problem of residual scaling is discussed.

In the method of multiple paired comparisons the dominance \( d_{ik}^j \) of object \( j \) over object \( k \) on attribute \( i \) \((i = 1, \cdots, p)\) is observed for each ordered pair in a finite class of objects. This numerical value of pairwise dominance is obtained from a graded paired comparison, and \( d_{ik}^j > 0, d_{ik}^j < 0, \) or \( d_{ik}^j = 0 \) indicating, respectively, the dominance of \( j \) over \( k \), \( k \) over \( j \), or an indifference between \( j \) and \( k \) on attribute \( i \). A general linear model and the resulting multivariate variance analysis of the set \( \{d_{ik}^j, \cdots, d_{ip}^j\} \) of vector observations has been given in a previous paper [Bechtel, 1968]. In that analysis these vectors are assumed to be independently \( p \)-variate normal with a common covariance matrix \( \Sigma \). However, within an ordered pair \( (j, k) \) the observations \( d_{ik}^j, \cdots, d_{ip}^j \) may be intercorrelated and measured in noncomparable units.

In the present paper the previous linear model is extended to include covariants upon which the \( p \) observation \( s \) are presumably dependent. These additional independent variables \( z_{ik}^j, \cdots, z_{ip}^j \) are incorporated into the extended linear model which forms the basis for the covariance adjustments of the preceding analysis. The need for the present extension is particularly great in situations in which \( d_{ik}^j \) represents a change in the dominance of \( j \) over \( k \) upon attribute \( i \), and \( z_{ik}^j \) represents the initial dominance of \( j \) over \( k \).

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from which this change occurred. However, more generally, the present analysis adjusts the variation in the vector \( d_{ik} = (d_{i1}^{j}, \ldots, d_{ip}^{j}) \) for the concomitant variation in the vector \( z_{ik} = (z_{i1}^{j}, \ldots, z_{ip}^{j}) \), where \( s \) may or may not equal \( p \), and where \( z_{ik} \) may or may not contain pairwise dominance values.

The univariate (\( p = 1 \)) or multivariate (\( p > 1 \)) analysis of covariance is especially suitable for treating residual variation in the observations after variation due to the concomitant variables has been removed. Rao [1965] regards covariance analysis as a method of testing for any "additional information" in \( (d_{i1}^{j}, \ldots, d_{ip}^{j}) \) beyond that contained in \( (z_{i1}^{j}, \ldots, z_{ip}^{j}) \). Indeed, if certain hypotheses are all retained following the covariance adjustments, then one is led to the conclusion that the vector \( d_{ik} \) is entirely dependent upon the vector \( z_{ik} \). The components of \( z_{ik} \) are analogous to factors in conventional factor analysis. That is, when these \( s \) concomitant variables have been "taken out" of the \( d_{ik} \), appropriate tests may be made for systematic variation in the residual data matrix. However, in covariance analysis, unlike factor analysis, the residual observations are imbued with an explicit linear structure for which rigorous estimation and testing procedures are available.

The next two sections are devoted to the layout \( \{d_{ik}\} \) of \( p \)-dimensional random vectors and the extended linear model for their analysis. Subsequently, the multivariate covariance analysis for this setup will be derived from the corresponding multivariate variance analysis [Bechtel, 1968]. As already indicated, the covariance extension yields adjustments for the parameter estimates and hypothesis tests of the variance analysis. However, it also provides additional inferential procedures for the \( s \) parameters associated with the concomitant variables. These estimation and testing procedures are illustrated with an analysis of political judgment data.

*The Layout*

Figure 1 illustrates the \( n \times n \) layout of ordered pairs for an experiment or a naturalistic study involving \( n \) psychological objects. This layout for the multiple paired comparisons design excludes the \( (j, j) \) cells \( (j = 1, \ldots, n) \). These diagonal observations have little meaning with many types of objects, and their exclusion implies that there are \( n^2 - n = n(n - 1) \) cells represented in a pairwise experiment. For each cell \( (j, k) \) in Figure 1 there is observed a \( p \)-dimensional random vector \( d_{ik} \) and a corresponding fixed vector \( z_{ik} \). Thus for the pairwise multivariate analysis of covariance the extended vector of observations will be written as

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(z_{ik}, d_{ik}) = (z_{i1}^{j}, \ldots, z_{ip}^{j}, d_{i1}^{j+1}, \ldots, d_{ip}^{j+p}),
\]

which is a partitioned row vector consisting of \( s + p \) measures associated with the ordered pair \( (j, k) \). In the present unreplicated pairwise design