ON THE CONSTRUCTION OF WEAK ORDERS FROM FRAGMENTARY INFORMATION*

PETER C. FISHBURN†
THE PENNSYLVANIA STATE UNIVERSITY

An iterative method is proposed for constructing a weak order from a partial order on a set of stimuli that is based on individual pairwise comparison data. The method generalizes Duncan Luce's construction of the weak order induced by a semiorder. Various aspects of the iterative procedure are discussed, including its rationale, the number of iterations required to obtain a weak order, and the extent to which the data support additions to the initial partial order as a function of the number of iterations performed before the additions occur.

1. Introduction

This paper discusses a natural iterative procedure for constructing a weak order on a set of stimuli from a partial order based on pairwise comparisons. We consider the type of situation in which the stimuli in a set X are, at least conceptually, weakly ordered according to some aspect or attribute, which might be physical (length, time of endurance, volume) or subjective (preferability, likelihood of occurrence). Individual pairwise-comparison judgments are to be used to identify the order. Our individual, limited as he is by perceptual and cognitive constraints, may be able to identify only a fragment of the weak order. The question then arises as to the possibility of recovering or reconstructing the weak order, or a sizable part of it, from the individual's fragmentary information. In as much as many weak orders may be consistent with the fragmentary information [Szpilrajn, 1930], the following alternative question may be more to the point: What weak order is most fully supported by the individual's fragmentary information?

As phrased in the latter question, this inference problem seems too generally stated to receive a specific answer. At the very least, one would want information on the nature of the process that generates the data, the type of judgment data obtained, and appropriate criteria for judging the goodness of fit of the data to (or the degree to which the data supports) a weak order. Rather than trying to attack this question in any depth, the present paper...
is limited to an examination of one particular way of constructing a weak order from fragmentary information.

The starting point for our constructive procedure is taken to be a strict partial order \( P \) on \( X \), so that \( P \) is asymmetric and transitive. The next section comments on how \( P \) might be obtained from an individual’s judgment data when it is not already in the form of a strict partial order. \( xPy \), which is an alternate way of writing \((x, y) \in P\), might mean that \( x \) is judged to be louder than \( y \), or \( x \) is preferred to \( y \), or \( x \) is judged to be more probable than \( y \).

From \( P \) we define the symmetric complement \( I \) on \( X \) by \( xIy \) if and only if \( \neg (xPy) \) and \( \neg (yPx) \). The relation \( I \) is variously referred to as similarity, indifference, matching, and so forth. In making judgments, the individual might indicate \( I \) as well as \( P \) directly: for example, if he feels reasonably sure that \( x \) has “more” of the focal attribute than does \( y \), he indicates \( xPy \); similarly for \( yPx \); and if he is not reasonably sure that one of \( x \) and \( y \) has “more” of the attribute than the other, he indicates \( xIy \).

When \( P \) is a strict partial order, its symmetric complement \( I \) is reflexive and symmetric but not necessarily transitive. This would be the case when \( xIy, yIz \) and \( xPz \), indicating perhaps that \( x \) and \( y \) lie within a discriminatory threshold, and likewise for \( y \) and \( z \), whereas there is a noticeable difference between \( x \) and \( z \) with respect to the focal attribute. When \( I \) is transitive, \( P \) will be called a strict weak order.

Equivalently, \( P \) is a strict weak order if and only if it is asymmetric and negatively transitive. The latter property means that for all \( x, y, z \) in \( X \), either \( xPz \) or \( zPy \) when \( xPy \). From this definition it is seen that negative transitivity eliminates the ability to detect a sensory or cognitive threshold from only the information provided by \( P \): if \( x \) is perceived to be different from \( y \) as regards the focal attribute, and if \( z \) is any other stimulus, then either \( z \) is perceived to be different from \( x \) or different from \( y \). And, when \( P \) is a strict weak order, if a noticeable difference is not perceived between \( x \) and \( y \), and if \( z \) is any other stimulus, then either a noticeable difference is not perceived between either \( x \) and \( z \) or \( y \) and \( z \), or else a noticeable difference is perceived both between \( x \) and \( z \) and between \( y \) and \( z \). As suggested earlier, we are most concerned with data for which \( I \) is not transitive.

The term “weak order” used earlier in this section is taken to mean a strict weak order as defined above. Beginning with a strict partial order \( P \) on \( X \), the procedure adds new ordered pairs to \( P \) in a series of steps until the augmented \( P \) turns into a strict weak order. To facilitate a description of the procedure we shall use the notion of relational products or compositions of binary relations. If \( R \) and \( S \) are binary relations on \( X \), then \( RS = \{(x, y) : xRz \text{ and } zSy \text{ for some } z \text{ in } X \} \). The binary relation \( RS \) is the composition of \( R \) and \( S \).

In the first step of the construction procedure we look for pairs \((x, y)\) in \( I \) for which there is a \( z \) in \( X \) such that either \( xIzPy \) or \( xPzIy \); that is, we look for \( xIy \) for which \( x(IP \cup PI)y \). The reason for this is easy to see. For example,