STANDARD ERRORS FOR OBLIQUELY ROTATED FACTOR LOADINGS*

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In a manner similar to that used in the orthogonal case, formulas for the asymptotic standard errors of analytically rotated oblique factor loading estimates are obtained. This is done by finding expressions for the partial derivatives of an oblique rotation algorithm and using previously derived results for unrotated loadings. These include the results of Lawley for maximum likelihood factor analysis and those of Girshick for principal components analysis. Details are given in cases including direct oblimin and direct Crawford-Ferguson rotation. Numerical results for an example involving maximum likelihood estimation with direct quartimin rotation are presented. They include simultaneous tests for significant loading estimates.

1. Introduction

In an earlier paper Archer and Jennrich [1973] derived formulas for the asymptotic standard errors of orthogonally rotated factor loading estimates. Corresponding results are derived here for obliquely rotated loadings.

We begin with a p by k factor loading matrix \( A = (a_{ir}) \) and an asymptotically normal estimate \( \hat{A} = (\hat{a}_{ir}) \). Maximum likelihood estimates in the classical factor analysis model [Lawley, 1967] and the principal components estimates in principal components analysis [Girshick, 1939] are both of this form. We are interested in the effect of an oblique rotation algorithm \( h \) on the asymptotic distribution of \( \hat{A} \). A function \( h \) is an oblique rotation algorithm if it maps an arbitrary p by k matrix \( X \) into a p by k matrix \( Y = XT \) where \( T \) is a nonsingular k by k matrix whose inverse has normalized rows, i.e.,

\[
\text{diag}(T'T)^{-1} = I,
\]

the k by k identity matrix. The value of \( T \) may, and generally will, be a function of \( X \). We are interested in the asymptotic distribution of the rotated

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loading estimates $\hat{A} = h(A) = \hat{A} \hat{T}$ and the rotated factor correlation estimates $\Phi = (\hat{T}'\hat{T})^{-1}$. Of special interest are the cases when $h$ represents oblimin [Harman, 1967, p. 324] or Crawford–Ferguson [1970] rotation.

As in previous work on the orthogonal case, the results derived here apply equally well to principal components analysis and to maximum likelihood factor analysis. Because of the work of Jöreskog [1967], Jennrich and Robinson [1969], and Clarke [1970] maximum likelihood factor analysis has become computationally feasible. In spite of this, however, principal components analysis is still too popular to be ignored.

Jöreskog [1969] has given standard error formulas for maximum likelihood loading estimates in the case when the rotation is determined by an apriori specification of a sufficient number of loading values. In contrast, our primary concern here is with analytic rotation.

2. Asymptotic Distributions under Oblique Rotation

Let $\Lambda = h(A) = (\lambda_{ir})$ denote the “true” rotated loadings and $\Phi = g(A) = (T'T)^{-1} = (\phi_{nr})$ the “true” rotated factor correlations. Assume that at $A$, $h$ has a differential $dh$ and $g$ has a differential $dg$. Then $dh$ is a linear transformation which maps $p$ by $k$ matrices into $p$ by $k$ matrices. Similarly $dg$ is linear and maps $p$ by $k$ matrices into symmetric $k$ by $k$ matrices and these linear maps approximate $h$ and $g$ at $A$. It follows from standard asymptotic theory [Rao, 1965, p. 321] that

\begin{equation}
\sqrt{n} (\hat{A} - A) \approx d(h(\sqrt{n} (\hat{A} - A))
\end{equation}

and

\begin{equation}
\sqrt{n} (\hat{\Phi} - \Phi) \approx d(g(\sqrt{n} (\hat{A} - A))
\end{equation}

where “$\approx$” means that the difference between the left- and right-hand sides of (1) and (2) approaches zero in probability as the sample size $n$ on which the estimate $\hat{A}$ is based approaches infinity. Since $dh$ and $dg$ are linear and $\hat{A}$ is an asymptotically normal estimate of $A$, $\hat{A}$ and $\hat{\Phi}$ are asymptotically normal estimates of $A$ and $\Phi$ whose asymptotic covariance matrices may be obtained from that of $\hat{A}$. In terms of the partial derivatives of $h$ and $g$,

\begin{equation}
acov (\lambda_{ir}, \lambda_{is}) = \Sigma_{mnxy} \frac{\partial h_{ir}}{\partial \alpha_{mx}} acov (\hat{\alpha}_{mx}, \hat{\alpha}_{ny}) \frac{\partial h_{is}}{\partial \alpha_{ny}}
\end{equation}

and

\begin{equation}
acov (\phi_{rs}, \phi_{uv}) = \Sigma_{mnxy} \frac{\partial g_{rs}}{\partial \alpha_{mx}} acov (\hat{\alpha}_{mx}, \hat{\alpha}_{nv}) \frac{\partial g_{uv}}{\partial \alpha_{nv}}
\end{equation}

Formulas for the asymptotic covariances $acov (\hat{\alpha}_{mx}, \hat{\alpha}_{nv})$ of the unrotated loadings on the right, i.e., the covariances in the limiting distribution of