Algorithm 19

Step-Cycle Generation

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Abstract — Zusammenfassung

Algorithm 19. Step-Cycle Generation. For given integer \( m > 1 \), each step-cycle corresponds to a set of permutations such that the step-cycles constitute a set of equivalence classes on the set of all permutations on \( m \) elements.

The algorithm has been used in connection with computations to search for groups consisting of a union of disjoint sets of permutations such that each set of permutations corresponds to a step-cycle, see [2] and [8].

Let \( m = n - 1 \) and \( n \geq 2 \) be an integer and \( P = (i_1) \), \( i = 1, 2, \ldots, n-1 \) a permutation on the set \( S = \{1, 2, \ldots, n-1\} \). The factorial representation of \( s, 0 \leq s < m! \) is

\[
s = (t[1] - 1)(m - 1)! + (t[2] - 1)(m - 2)! + \ldots + (t[m - 1] - 1) \cdot 1!
\]

where \( 1 \leq t[i] \leq m + 1 - i, \ i = 1, 2, \ldots, m - 1 \), see [1].

In [7], see also [5] and [6], Pager defines an \( m \)-ary \( p \)-number as the sequence \( t[1] \ldots t[m] \) where \( t[m] = 1 \). When the permutations are ordered lexicographically there is a one-to-one correspondence between the permutation \( P \) and the \( p \)-number \( T(P) = t[1] \ldots t[m] \) such that the ordering number \( N(P) \), where \( 1 \leq N(P) \leq m! \) corresponds to \( T(P) \). For \( P = (i) \) then \( N(P) = 1 \) and \( T(P) = 11 \ldots 1 \). The complementary permutation to \( P \) is \( \bar{P} = (a_{n-1}) \). In my paper [3] it was proved that the sum of the ordering numbers of two complementary permutations is \( N(P) + N(\bar{P}) = (n - 1)! + 1 \).
A step-cycle $D = (d_1 d_2 \ldots d_n) = (d_i)$ is a cycle of $n$ positive integers, $0 < d_i < n$, which satisfy the following conditions:

(i) $\sum_{i=1}^{n} d_i \equiv 0 \pmod{n}$

(ii) By “accumulation” from one arbitrary starting point, all the residue classes modulo $n$ are represented once each (Selmer [8]).

Because of (i), property (ii) then holds for all $n$ starting points. The $n-1$ first residue classes by each accumulation represent a permutation of the residue classes $1, 2, \ldots, n-1$, that is, a permutation on $S$. A step-cycle $D$ gives rise to a set $\mathbb{D}$ of different permutations. These sets of permutations on $S$ form a set of equivalence classes on the set of all permutations on $S$.

The permutation $P$ is said to generate the step-cycles $D$ and $\bar{D} = (n-d_i)$ if $N(P) = \min \{N(P_i); P_i \in D \text{ or } P_i \in \bar{D}\}$. Hence, the algorithm gives the step-cycles in complementary pairs ordered lexicographically after their generating permutations. It can be proved that $D = \bar{D}$ (as cycles) if and only if $n$ is even and $d_i = n - d_{n+1-i}$, see [4]. Another important observation is that for $P \in \mathbb{D}$ then $P \in \bar{\mathbb{D}}$.

The table lists all step-cycles for $3 \leq n \leq 6$ as $n$-digit numbers.

<table>
<thead>
<tr>
<th>$n=3$</th>
<th>$n=4$</th>
<th>$n=5$</th>
<th>$n=6$</th>
</tr>
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<tbody>
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<td>111</td>
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<td>11111</td>
<td>111111</td>
</tr>
<tr>
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<td>3333</td>
<td>44444</td>
<td>555555</td>
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<td>311214</td>
</tr>
<tr>
<td>522144</td>
<td>144144</td>
<td>522522</td>
<td>443223</td>
</tr>
</tbody>
</table>

begin integer $i, j, c, n, m$;
integer array $a[1:10], b[1:10], d[1:11], e[1:11], g[1:10], t[1:10]$;
boolean first;
procedure perm($b, m$);
integer array $b$; integer $m$;
begin integer $i, j$;
if first EQIV false then go to r3;
for $i := 1$ step 1 until $m$ do $t[i] := 1$;
first := false;
g[1]:=(m+1)//2; g[2]:=(m-1)//2;