CORRELATION AND PREDICTION IN ORDINAL TEST THEORY

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Based on the test theory model for ordinal measurements proposed by Schulman and Haden, the present paper considers correlations between tests, attenuation, regressions involving true and observed scores, and prediction of test reliability.

The population correlation between tests is shown to be related to the expected sample correlation for samples of size $n_1$ and $n_2$. Errors of estimation, measurement and prediction are found to be similar to their counterparts in interval test theory, while attenuation is identical to its counterpart. The bias in estimating population reliability from sample data is compared for Kendall's tau and Spearman's rho.

Key words: attenuation, error of estimation, error of measurement, error of prediction, reliability estimation.

1. Introduction

Schulman and Haden [1975] have proposed a test theoretic model for ordinal data. For a finite population of $N$-persons, the $N!$ possible orderings are assumed to have probabilities $p_j, j = 1, \cdots, N!$. Such a test is denoted by $\Theta(N, p)$. Letting $R_j(X_i), i = 1, \cdots, N, j = 1, \cdots, N!$, represent the rank of the $i$th individual on the $j$th ordering, the true score of the $i$th individual is defined as his expected rank,

$$T_i = \varepsilon_i R_j(X_i) = \sum_{j=1}^{N!} R_j(X_i)p_j .$$

Defining an error score as $E_{ij} = R_i(X_i) - T$, yields the model

$$R_j(X_i) = T_i + E_{ij} .$$

The generic reliability of such a test is defined as $\rho_{RT}^2$ and is known to be equal to $\sigma_T^2/\sigma_R^2$.

Now, letting $D_n$ denote the set of $\binom{N}{n}$ samples of size $n$ from the $N$-population, each choice of $m \in D_n$ results in a new ordinal test $\Theta_m(n, p)$. The model for the new test may be written as

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where \( r_{jm}(X_i) \) is the rank of individual \( i \) within the \( m \)th \( n \)-sample on ordering \( j \), \( t_{jm} = \sum_i r_{jm}(X_i) \) is his specific true score and \( e_{jm} \) is the corresponding error term. Since \( \varepsilon_n(n, p) \) is also an ordinal test, its specific reliability, \( \rho_{n}^2 = \rho_{r_t}^2 \), must necessarily be equal to \( \sigma_t^2 / \sigma_r^2 \). The expected specific reliability over random sampling of \( n \) from \( N \) is denoted by \( \varepsilon_n \rho_{n}^2 \).

Schulman and Haden derived a relationship involving the expected specific reliabilities for \( n_1 \)- and \( n_2 \)-samples, and the generic reliability, \( \rho_{RT}^2 \). In Section 2 of the present paper, an analogous relationship is developed for correlations between two ordinal tests. Section 3 presents a brief note on attenuation for ordinal tests. In Section 4, errors of estimation, measurement and prediction are shown to be similar to their counterparts in interval test theory. Finally, Section 5 briefly discusses the use of Kendall’s tau and Spearman’s rho in estimating reliability.

2. Correlations Between Tests

Consider two distinct ordinal tests, \( \Theta(N, p) \) and \( \Theta'(N, p') \), on the same \( N \)-population; that is, \( R = T + E \) on \( \Theta \) and \( R' = T' + E' \) on \( \Theta' \). The generic correlation will be denoted by \( \rho_{RR} \) and the specific correlation for the \( m \)th \( n \)-sample will be denoted by \( \rho_{rr}(m) \). We note that the estimation of \( \rho_{RR} \) is essential for validity studies. Furthermore, if only a randomly chosen \( n \)-sample is available for observation, then this generic correlation must be estimated from sample data. To this end we must relate the expected specific correlation \( \varepsilon_n \rho_{rr}(m) \) to the generic correlation \( \rho_{RR} \). It will be shown that a result similar to the corollary to Theorem 3 of Schulman and Haden holds for these intertest correlations. Specifically, we shall show that for \( 1 \leq n_1, n_2 \leq N \),

\[
\rho_{RR} - \varepsilon_n \rho_{rr}(b) = \frac{(n_1 + 1)(N - n_2)}{(n_2 + 1)(N - n_1)} \left[ \rho_{RR} - \varepsilon_n \rho_{rr}(a) \right],
\]

where \( a \in D_{n_1} \), \( b \in D_{n_2} \).

To see this we define a joint propensity distribution for \( \Theta \) and \( \Theta' \). Let

\[
q_{jk} = P(\text{ordering } R_j \text{ on } \Theta \text{ and ordering } R_k' \text{ on } \Theta'),
\]

\( j, k = 1, \ldots, N! \). Regardless of the form of the joint distribution, the marginals are still \( p \) and \( p' \). That is,

\[
\sum_{i=1}^{N!} q_{ik} = p_i \quad \text{and} \quad \sum_{i=1}^{N!} q_{ik} = p_k'.
\]

Employing this framework, the proof of (1) follows the same pattern as the proof of Schulman and Haden’s Theorem 3, with one exception. Expectations are taken over the joint propensity distribution, \( \varepsilon_q \), rather than over the two-dimensional propensity for a single test, \( \varepsilon_{pq} \). An alternative proof of (1) can be constructed by employing equation (6) of Durbin and Stuart [1951],