RANGE RESTRICTIONS FOR PRODUCT-MOMENT CORRELATION MATRICES

INGRAM OLKIN
STANFORD UNIVERSITY

It is well-known that for a trivariate distribution if two correlations are fixed the remaining one is constrained. Indeed, if one correlation is fixed, then the remaining two are constrained. Both results are extended to the case of a multivariate distribution. The results are applied to some special patterned matrices.

Key words: constrained correlations, multiple correlation, partial correlation, patterned correlation matrices.

It is well-known that if a subset of correlations is given, that the ranges of the remaining correlations are restricted. Suppose that the correlation matrix \( R \) of \( p \) variables \( X = (X^{(1)}, X^{(2)}, X^{(3)}) \), where \( X^{(i)} \) is of order \( p_i \), \( p_1 + p_2 + p_3 = p \), is given in partitioned form by

\[
R = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\]

The problem is to find restrictions on a submatrix of \( R \) when the complementary submatrices are given. Two distinct subproblems emerge: (i) the case where all submatrices but \( R_{11} \) are fixed, (ii) the case where all submatrices but \( R_{12} \) are fixed.

The trivariate case \( p = 3 \) \((p_1 = 1, p_2 = 2)\) has received considerable attention. If \( r_{13} \) and \( r_{23} \) are given, then \( r_{12} \) is restricted by the inequalities

\[
(1 - r_{12}^2)(1 - r_{23}^2)^{1/2} \leq r_{12} \leq r_{13} r_{23} - [(1 - r_{13}^2)(1 - r_{23}^2)]^{1/2}.
\]

McCornack [1956] discusses the use of (2) when different weighing methods are applied to a set of tests or test items. Stanley and Wang [1969] note that condition (2) is obtained from the fact that the partial correlation \( r_{12.3} \) is in the interval \([-1, 1]\). Actually the two conditions are equivalent, and this equivalence is useful in the more general case. Novel geometric proofs of (2) are provided by Glass and Collins [1970] and by Leung and Lam [1975].

As a first extension consider the case \( p_1 = 1, p_2 = p - 1 \). If,

\[
R = \begin{pmatrix} 1 & r_{12} & \cdots & r_{1p} \\ r_{22} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ r_1 \\ R_{22} \end{pmatrix},
\]

and \( R_{22} \) is given, then how free is \( r_1 \)? The answer is provided by Hubert [1972] in the form of a quadratic inequality:

\[
r_1 R_{22}^{-1} r_1^t \leq 1.
\]

A more intuitive interpretation of (4) is the equivalent statement that the squared multiple

Reprint requests should be addressed to Ingram Olkin, Sequoia Hall, Department of Statistics, Stanford University, Stanford, California 94305.
correlation satisfies

\[ r_{i(2, \ldots, p)}^2 \leq 1. \]  

Hubert's [1972] proof of (4) is based on the positive-definiteness of \( R \). Indeed, this condition is fundamental in more general contexts.

To resolve case (i) consider

\[
R = \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix}
\]

with \( R_{12} \) and \( R_{22} \) given. Positive-definiteness of \( R \) is equivalent to positive-definiteness of \( R_{22} \) and the condition

\[ R_{11} > R_{12}R_{22}^{-1}R_{21}, \]  

where \( A > B \) means that the matrix \( A - B \) is positive definite.

In the particular case that \( R_{12} = (1 \quad q_{12} \quad q_{12}) \)

condition (6) is equivalent to the inequalities

\[ q_{12} \leq [(1 - q_{11})(1 - q_{22})]^{1/2} \leq r_{12} \leq q_{12} + [(1 - q_{11})(1 - q_{22})]^{1/2}, \]  

where

\[
Q = \begin{pmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{pmatrix} = \begin{pmatrix}
r_{13} & \cdots & r_{1p} \\
r_{23} & \cdots & r_{2p}
\end{pmatrix} \begin{pmatrix}
R_{12} \quad R_{22} \\
R_{31} \quad R_{33}
\end{pmatrix}^{-1},
\]

and \( R_{(3)} = (r_{ij}), \quad i, j = 3, 4, \ldots, p \). As in the trivariate case, inequalities (7) are equivalent to the assertion that the partial correlation \( r_{12(3, \ldots, p)} \) must lie in the interval \([-1, 1]\).

The resolution of case (ii) is more cumbersome. In (1) all matrices but \( R_{12} \) are fixed. For simplicity of notation, write \( Z \) for \( R_{12} \). Positive-definiteness of \( R \) implies positive-definiteness of

\[
\begin{pmatrix}
R_{11} & Z \\
Z' & R_{22}
\end{pmatrix} - \begin{pmatrix}
R_{13} & R_{32} \\
R_{32} & R_{33}
\end{pmatrix} R_{33}^{-1} (R_{31} \quad R_{32})
\]

which is equivalent to

\[ R_{22} > (Z - R_{13} R_{33}^{-1} R_{32}) R_{11}^{-1} (Z - R_{13} R_{33}^{-1} R_{32}), \]  

or

\[ R_{11} > (Z - R_{13} R_{33}^{-1} R_{32}) R_{22}^{-1} (Z - R_{13} R_{33}^{-1} R_{32}), \]  

where

\[ R_{jj} = R_{jj} - R_{j3} R_{33}^{-1} R_{3j}, \quad j = 1, 2. \]

In the special case that \( p_1 = 1 \), so that \( R_{11} = 1 \), condition (10) reduces to a quadratic inequality that is a generalization of (4).