SOME CLARIFICATIONS OF THE CANDECOMP ALGORITHM APPLIED TO INDSCAL

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Carroll and Chang have claimed that CANDECOMP applied to symmetric matrices yields equivalent coordinate matrices, as needed for INDSCAL. Although this claim has appeared to be valid for all practical purposes, it has gone without a rigorous mathematical footing. The purpose of the present paper is to clarify CANDECOMP in this respect. It is shown that equivalent coordinate matrices are not granted at global minima when the symmetric matrices are not Gramian, or when these matrices are Gramian but the solution not globally optimal.

Key words: CANDECOMP, PARAFAC, INDSCAL.

Carroll and Chang (1970) and Harshman (1970) have independently suggested the same method of analyzing three-way arrays and christened this method CANDECOMP and PARAFAC, respectively. If $Z$ is a $p \times q \times m$ three-way array containing $m$ frontal slabs $Z_1, \ldots, Z_m$, CANDECOMP/PARAFAC seeks to minimize the function

$$f(X, Y, D_1, \ldots, D_m) = \sum_{i=1}^{m} \|Z_i - XD_i Y\|^2, \quad (1)$$

where $X$ is a $p \times r$ matrix, $Y$ is a $q \times r$ matrix, $D_i$ is a diagonal $r \times r$ matrix, and $r$ is a fixed rank-parameter. Carroll and Chang (1970) also considered the function

$$g(X, D_1, \ldots, D_m) = \sum_{i=1}^{m} \|S_i - XD_i X^*\|^2, \quad (2)$$

where, for $i = 1, \ldots, m$, $S_i$ is a given symmetric $p \times p$ matrix, and $X_i$ and $D_i$ are as in (1). This function is to be minimized in the well-known INDSCAL method. To minimize (2), Carroll and Chang suggested using CANDECOMP, and justified this by claiming that, when the CANDECOMP process (applied to $S_1, \ldots, S_m$) finally converges, $X$ and $Y$ will be equivalent in the sense that their columns will be equal up to scalar multiplication. More recently, the claim has been repeated by Carroll and Pruzansky (1984), among others.

Ten Berge, Kiers, and de Leeuw (1988) have shown that, for a contrived set of matrices, nonequivalence may hold at certain accumulation points of the CANDECOMP/PARAFAC process. Their result, however, is not incompatible with the equivalence claim of Carroll and Chang, because $f$ has no minimum, and hence CANDECOMP does not converge for the data set and rank ($r = 2$) they considered. Practical experience with CANDECOMP has shown that equivalence is indeed guaranteed for all practical purposes. However, mathematical proofs for equivalence have been absent.

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The purpose of the present paper is to examine equivalence from a mathematical point of view, as a further clarification of CANDECOMP applied to INDSCAL.

The organization of the present paper is as follows. First, the phenomenon of equivalence ($X$ and $Y$ proportional columnwise) is related to symmetry of the matrices $XD_i Y'$. Symmetry is necessary and in most cases sufficient for equivalence; hence, examining conditions for symmetry is relevant for examining equivalence. Next, we consider the case $m = 1$ for (1) and (2), and show that asymmetry is possible in the case of certain indefinite matrices. This permits the construction of cases where CANDECOMP has asymmetric solutions, when applied to indefinite matrices $S_1, \ldots, S_m$, for $m > 1$. Finally, the case where $S_1, \ldots, S_m$ are Gramian (nonnegative definite) is treated. Surprisingly, it is shown that the CANDECOMP function does have stationary points where nonequivalence holds. On the other hand, equivalence can be shown to hold at the global minimum of the CANDECOMP function if $r = 1$ or if $X$ and $Y$ are constrained to be columnwise orthonormal. Neither a proof nor a counterexample to symmetry have been found for $r > 1$ and $X$ and $Y$ unconstrained, at the global minimum of the CANDECOMP function.

From Equivalence to Symmetry

When CANDECOMP is applied to symmetric matrices $S_1, \ldots, S_m$, upon convergence we obtain “regression” matrices

$$
\hat{S}_i = XD_i Y',
$$

for $i = 1, \ldots, m$. If a CANDECOMP solution is to be of use for INDSCAL, $X$ must equal $Y$, or at least $X$ and $Y$ should be proportional columnwise (equivalent). It is important to note that equivalence is directly related to symmetry of the regressions, henceforth referred to as “symmetry”. Results 1 and 2 below pinpoint this relationship.

**Result 1.** Equivalence is sufficient for symmetry.

**Proof.** Trivial. □

Harshman (1972) has shown that, given the regression matrices $\hat{S}_1, \ldots, \hat{S}_m$, the set of matrices $X$, $Y$ and $D_1, \ldots, D_m$ that satisfy (3) is unique up to certain permutations and scalar multiplications, provided that at least one pair $D_i, D_j$ satisfies the conditions that they are nonsingular, all diagonal elements of $D_i D_j^{-1}$ are distinct, and $X$ and $Y$ have full column rank. The latter conditions, referred to as the “uniqueness conditions”, also play a role in the next result.

**Result 2.** Equivalence is necessary for symmetry if the uniqueness conditions of Harshman are satisfied.

**Proof.** Let it be assumed that symmetry holds, and that for a pair $D_i$ and $D_j$, both nonsingular, the diagonal elements of $D_i D_j^{-1}$ are distinct. Then, we have

$$
XD_i Y' = YD_i X',
$$

and

$$
XD_j Y' = YD_j X'.
$$

Because $X$ and $Y$ span the same column-space, $X = YT$ for some nonsingular matrix $T$, and hence,