ALTERNATIVE SOLUTIONS FOR OPTIMIZATION PROBLEMS IN GENERALIZABILITY THEORY

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Solutions for the problem of maximizing the generalizability coefficient under a budget constraint are presented. It is shown that the Cauchy-Schwarz inequality can be applied to derive optimal continuous solutions for the number of conditions of each facet.

Key words: generalizability theory, Cauchy-Schwarz inequality, optimal designs.

Introduction

A procedure for the problem of maximizing the generalizability coefficient, \(\rho^2\), of the two-facet random-model crossed design under a budget constraint was recently presented by Sanders, Theunissen, and Baas (1991), who formulated this optimization problem as

\[
\begin{align*}
\text{minimize} & \quad \frac{\sigma^2_{p1}}{n_1} + \frac{\sigma^2_{p2}}{n_2} + \frac{\sigma^2_{\text{res}}}{n_1n_2} = \nu, \\
\text{subject to} & \quad c_1n_1 + c_2n_2 + c_{12}n_1n_2 \leq c, \quad \text{and} \\
& \quad n_1 \text{ and } n_2 \text{ integer } \geq 1. 
\end{align*}
\]

Maximizing the generalizability coefficient is equivalent to minimizing the error variance, that is, the sum of the three interaction variance components. In (1), \(\sigma^2_{p1}\) is the variance component for the person by Facet 1 interaction, \(\sigma^2_{p2}\) is the variance component for the person by Facet 2 interaction, \(\sigma^2_{\text{res}}\) is the variance component for the person \(\times\) Facet 1 \(\times\) Facet 2 interaction plus error, and \(n_1\) and \(n_2\) are the number of conditions of Facet 1 and Facet 2. The minimization statement in (1) expresses that the value of the objective-function is determined by the numbers of conditions, \(n_1\) and \(n_2\), used for Facet 1 and 2.

In the cost function in (2), the right-hand term specifies an upper limit on the budget. The costs required by the conditions of Facet 1 are specified by the term \(c_1n_1\), \(c_1\) being the cost of one condition of Facet 1. The costs required by the conditions of Facet 2 are specified by the term \(c_2n_2\), \(c_2\) being the cost of one condition of Facet 2. The number of observations per subject equals \(n_1n_2\). Thus, denoting the cost of one observation for the sample of subjects to be tested by \(c_{12}\), the costs necessary for the total number of observations are specified by the term \(c_{12}n_1n_2\).

The lower bound integer constraint (3) states that feasible values for \(n_1\) and \(n_2\) have to be integer values and each facet has to have at least one condition.
In the two-step procedure proposed by Sanders et al. (1991), optimal continuous solutions for \( n_1 \) and \( n_2 \) are derived for a continuous relaxation of (3) in the first step. These continuous solutions are then used as the bounds in a branch-and-bound algorithm to obtain the optimal integer solutions. The use of the Lagrange multipliers method to attain the optimal continuous solutions, however, can result in complex derivations. But, as first shown by Stuart (1954) for problems of sample survey theory, derivations of optimum solutions can be simplified if the Cauchy-Schwarz inequality applies. This inequality states that for any sets of real numbers

\[
\{a_h\}, \{b_h\}, \quad (h = 1, 2, \ldots, p),
\]

\[
(\sum a_h^2)(\sum b_h^2) \geq (\sum a_h b_h)^2,
\]  \hspace{1cm} (4)

with equality occurring if and only if \( a_h/b_h = k \) for all \( h \) and some constant \( k \). According to Stuart (1954, p. 239), the sampling variance in most of the problems in sample survey theory takes the form \( \sum (v_h/n_h) = \nu \), where \( v_h \) is the function of population parameters only, and \( n_h \) is a function of sample numbers only. The cost functions generally considered are of the form \( \sum n_h c_h = c \), where the \( c_h \) are fixed cost constants. From (4) it therefore follows that

\[
v c = \left( \sum \frac{v_h}{n_h} \right) \left( \sum n_h c_h \right) \geq \left( \sum (v_h c_h)^{1/2} \right)^2.
\]  \hspace{1cm} (5)

Since the right-hand side of (5) is independent of the \( n_h \), the minimization of \( vc \) for fixed \( c \) (or for fixed \( \nu \)) and variation in the \( n_h \) is achieved when, from (4) and (5),

\[
\frac{n_h^2 c_h}{v_h} = k^2 \quad (\text{all } h).
\]  \hspace{1cm} (6)

In the survey sampling literature (e.g., Cochran, 1977), applications of the Cauchy-Schwarz inequality to optimization problems abound. The purpose of the present note is to show its applicability to optimization problems recurring in generalizability theory.

**Solutions for Two-Facet Optimization Problems**

Other linear cost functions besides (2) can arise if \( c_1, c_2 \), or \( c_{12} \) is equal to zero. Functions with \( c_1 \) or \( c_2 \) and \( c_{12} \) equal to zero, however, lead to trivial optimization problems. Moreover, the cost function with \( c_1 \) and \( c_2 \) equal to zero, that is, cost function \( (c/c_{12}) \leq n_1 n_2 \), reduces the problem of optimum allocation for fixed cost to the problem of optimum allocation for a fixed number of observations. By replacing the inequality constraint in the cost function by an equality constraint, the optimal continuous solutions for this optimization problem,

\[
n_1 = \left( \frac{\sigma_{p1}^2}{\sigma_{p2}^2} \cdot \frac{c}{c_{12}} \right)^{1/2}, \quad \text{and} \quad n_2 = \left( \frac{\sigma_{p2}^2}{\sigma_{p1}^2} \cdot \frac{c}{c_{12}} \right)^{1/2},
\]

were derived by Woodward and Joe (1973). Therefore, three other potential linear cost functions can be distinguished:

\[
c_1 n_1 + c_{12} n_1 n_2 \leq c, \quad (7)
\]

\[
c_2 n_2 + c_{12} n_1 n_2 \leq c, \quad (8)
\]