TESTING EQUALITY OF CORRELATED PROPORTIONS WITH INCOMPLETE DATA ON BOTH RESPONSES

DINESH S. BHOJ
RUTGERS UNIVERSITY

TOM A. B. SNIJDERS
UNIVERSITY OF GRONINGEN

Two test statistics are proposed for testing the equality of two correlated proportions when some observations are missing on both responses. The performance of these tests in terms of size and power is compared with other tests by means of Monte Carlo simulations. The proposed tests are easily computed and compare favorably with other tests.

Key words: combination of tests, equality of correlated proportions, incomplete data, asymptotically most powerful test, Monte Carlo study, antithetic variates, power comparison.

Introduction

One of the problems in practical statistics is the comparison of two proportions when the observations on two related groups or on the same group under two different experimental conditions are available. When the data are complete McNemar's (1947) test is usually used for testing the hypothesis of the equality of two correlated proportions. There are many situations where the data are incomplete, that is, in addition to some complete pairs of observations on the first response only and some observations on the second response only, are recorded. We wish to use all the available data to test the hypothesis on the correlated proportions.

Campbell (1984) proposed a method to compute the likelihood ratio (LR) statistic using the EM-algorithm. The power of this LR test was studied by Gokhale and Sirotnik (1984). Ekbohm (1982) and Choi and Stablein (1982) proposed several hand calculable test statistics for this problem. The purpose of the present paper is to propose two tests which combine good power properties with ease of application. One test is an optimal combination of a test statistic for the paired (complete) data, and a test statistic for the unpaired (incomplete) data. The other test is just like the LR test asymptotically uniformly most powerful when the one-sided testing problem is considered, but it is less robust against violation of the assumption that the complete and the incomplete data are governed by the same probabilities. The tests are compared by Monte Carlo simulations.

Suppose that a dichotomous response is observed on n experimental units before and after a certain treatment, and in addition m_b units are observed only before and m_a units only after the treatment. Outcomes are labeled 0 and 1. The observed data can be represented in the following two fourfold tables.
It is assumed that the probabilities governing the complete and the incomplete data are the same, and that the mechanisms causing incomplete data are independent of the outcomes of the trials; all trials are assumed to be independent. The vector $\mathbf{N} = (N_{00}, N_{01}, N_{10}, N_{11})$ is then multinomially distributed with parameters $n$ and $(p_{00}, p_{01}, p_{10}, p_{11})$ while $M_{b1}$ and $M_{a1}$, respectively, are binomially distributed with parameters $m_b$ and $p_b$ and, respectively, $m_a$ and $p_a$; $N$, $M_{b1}$ and $M_{a1}$ are independent. The parameters are related by

$$P_b = p_{10} + p_{11}, \quad P_a = p_{01} + p_{11},$$

and, of course, $p_{00} + p_{01} + p_{10} + p_{11} = 1$. The null hypothesis is $H_0$: $p_b = p_a$, which is equivalent to $p_{10} = p_{01}$. The tested parameter is denoted by $\delta = p_a - p_b = p_{01} - p_{10}$, and $m = m_b + m_a$.

**Test Statistics**

If unpaired data are discarded, the usual procedure for testing $H_0$ is based on McNemar's (1947) test statistic

$$Z_0 = \frac{N_{01} - N_{10}}{(N_{01} + N_{10})^{1/2}},$$

which has asymptotically (for $n \to \infty$) under $H_0$ the standard normal distribution. If paired data are discarded (e.g., if $n$ is small relative to $m_b$ and $m_a$), the usual procedure is to use Fisher's exact test or the chi-squared test. An asymptotic approximation to Fisher's test and the one-sided analogue of the chi-squared test both result in the test statistic

$$Z_1 = \frac{\hat{p}_a - \hat{p}_b}{\{(m_b^{-1} + m_a^{-1})\hat{p}_a(1 - \hat{p}_a)\}^{1/2}},$$

where $\hat{p}_b = M_{b1}/m_b$, $\hat{p}_a = M_{a1}/m_a$ and $\hat{p}_a = M_{a1}/m$ is the estimator for the common success probability $p_b = p_a$ under $H_0$. This statistic has also asymptotically (for $m_b \to \infty$, $m_a \to \infty$) under $H_0$ the standard normal distribution.

The linear combination

$$Z_2 = \frac{(1 - \lambda)Z_0 + \lambda Z_1}{\{(1 - \lambda)^2 + \lambda^2\}^{1/2}},$$

for an appropriate value of $\lambda (0 < \lambda < 1)$, uses all available data. This type of statistic was proposed by Bhoj (1978, 1979) for testing hypotheses on the means and variances of a bivariate normal distribution with incomplete data. For $\lambda = 0$ or 1, respectively, it reduces to $Z_0$ or $Z_1$. The expectation of $Z_2$ under the alternative hypothesis is approximately

$$\delta\{(1 - \lambda)\hat{m}^{1/2}(p_{01} + p_{10})^{-1/2} + \lambda\hat{m}^{1/2}(p_a(1 - p_a))^{-1/2}\} \quad \{(1 - \lambda)^2 + \lambda^2\}^{1/2},$$

where $\hat{m} = m_b m_a / m$ and $p_a = (m_b p_b + m_a p_a) / m$. For small positive $\delta$, the power of the test based on $Z_2$ is approximately maximized by choosing $\lambda$ so that this expression is maximal. This is accomplished for