LOWER BOUNDS FOR THE RELIABILITY OF THE TOTAL SCORE ON A TEST COMPOSED OF NON-HOMOGENEOUS ITEMS:
II: A SEARCH PROCEDURE TO LOCATE THE GREATEST LOWER BOUND

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Finding the greatest lower bound for the reliability of the total score on a test comprising \( n \) non-homogenous items with dispersion matrix \( \Sigma_X \) is equivalent to maximizing the trace of a diagonal matrix \( \Sigma_E \) with elements \( \theta_i \), subject to \( \Sigma_E \) and \( \Sigma_T = \Sigma_X - \Sigma_E \) being non-negative definite. The cases \( n = 2 \) and \( n = 3 \) are solved explicitly. A computer search in the space of the \( \theta_i \) is developed for the general case. When Guttman's M (maximum split-half coefficient alpha) is not the g.l.b., the maximizing set of \( \theta_i \) makes the rank of \( \Sigma_T \) less than \( n - 1 \). Numerical examples of various bounds are given.

Key words: reliability bounds, non-homogeneous composites, search procedure, trace minimization with constraints.

1. Introduction

In the companion paper it was shown that the observed-score dispersion matrix \( \Sigma_X \) (assumed known or well-estimated) of the \( n \) items making up a test is the sum of the true-score and error-score dispersion matrices \( \Sigma_T \) and \( \Sigma_E \), where \( \Sigma_E \) is a diagonal matrix with elements denoted by \( \theta_i \). Thus if \( \Sigma_X \) has elements \( \sigma_{ij} \), then \( \Sigma_T \) has known off-diagonal elements \( \sigma_{ij} \) and unknown diagonal elements \( t_i = \sigma_{ii} - \theta_i \). It was also shown that finding the greatest lower bound (g.l.b.) for the reliability \( \rho \) of the total score on the test is equivalent to maximizing the trace \( \sum \theta_i \) of \( \Sigma_E \), or minimizing the trace \( \sum t_i \) of \( \Sigma_T \), subject to the conditions

(A) \( \Sigma_T \) is non-negative definite, and
(B) \( \theta_i \geq 0 \) (or \( t_i \leq \sigma_{ii} \)) for all \( i \):

the g.l.b. is the corresponding value of \( 1 - \sum \theta_i / \sigma_X^2 \), where \( \sigma_X^2 = \sum \sum \sigma_{ij} \).

Continuity considerations show that the g.l.b. will be attainable (that is, will be

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a possible value of $\rho$), and that the corresponding matrix $\Sigma_T$ will be positive semi-definite, with rank $n - 1$ or less, and with $\det \Sigma_T = 0$.

The present paper develops a method for finding the g.l.b. by search in the space of vectors $\theta = (\theta_1, \theta_2, \ldots, \theta_n)'$, referred to as the $\theta$-space, or in the space of vectors $t = (t_1, t_2, \ldots, t_n)'$, called the $t$-space. Since $t_i = \sigma_{ii} - \theta_i$, the $t$-space can be obtained from the $\theta$-space by reflection in the origin and a known translation; it is therefore merely a matter of convenience which space we use. In Sections 2 and 4 the cases $n = 2$ and $n = 3$ are solved explicitly, using $\theta$-space and $t$-space respectively. In Section 3 a general formulation of the search problem is given, and in Section 5 a computer search procedure is described which can, subject to constraints of machine size, handle any number of items.

Guttman's [1945] bounds are referred to at various points for comparison purposes: they were reviewed in the present framework in the companion paper.

In Section 6 we show that his $\lambda_4$ (sometimes) fails to be the g.l.b. only because the $\theta^*_i$ can be chosen so as to make the rank of $\Sigma_T$ less than $n - 1$. (Here, as in the companion paper, we use $\lambda_4$ to denote the largest split-half coefficient alpha, which is the bound Guttman implicitly proposed, although in his paper he uses the symbol $\lambda_4$ to denote any split-half coefficient alpha.) Numerical examples of his bounds, together with the g.l.b. are given in Section 4.

The expression $\text{diag} \ (a)$, where $a$ is a vector, denotes a diagonal matrix whose elements are, in order, those of $a$; for example $\Sigma_E = \text{diag} \ \theta$. Also $I$ denotes the vector $(1, 1, \cdots, 1)'$, and $I_n$ the identity matrix of order $n$.

2. The Case $n = 2$

If we work in $\theta$-space, the requirement that $\Sigma_T$ and $\Sigma_E$ be non-negative definite is equivalent to the inequalities

\begin{equation}
0 \leq \theta_1 \leq \sigma_{11}, \quad 0 \leq \theta_2 \leq \sigma_{22}
\end{equation}

and

\begin{equation}
(\sigma_{11} - \theta_1)(\sigma_{22} - \theta_2) \geq \sigma_{12}^2.
\end{equation}

Inequalities (2.1) confine $\theta$ to a rectangle OABC (see Fig. 1). If we take the equality sign in (2.2), we have a rectangular hyperbola with asymptotes $\theta_1 = \sigma_{11}$, $\theta_2 = \sigma_{22}$. Only one branch passes through the rectangle defined by (2.1), and the points satisfying (2.2) lie on the side of it nearer the origin. Thus (2.1) and (2.2) define an admissible region (shaded in Fig. 1) lying in the first quadrant between the axes and the hyperbola.

The equation $\theta_1 + \theta_2 = c$, a constant, represents a line of slope $-1$. If the tangent to the hyperbola with slope $-1$ meets it at a point $S$ of the admissible region, as in Fig. 1, the co-ordinates of this point give the maximum of $\theta_1 + \theta_2$.