A NOTE ON THE DETERMINATION OF CONFIGURATION AND WEIGHTS
FOR A CLASS OF INDIVIDUAL SCALING MODELS

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Consider the typical problem in individual scaling, namely finding a common configuration and weights for each individual from the given interpoint distances or scalar products. Within the STRAIN framework it is shown that the problem of determining weights for a given configuration can be posed as a standard quadratic programming problem. A set of necessary conditions for an optimal configuration to satisfy are given. A closed form expression for the configuration is obtained for the one dimensional case and an approach is given for the two dimensional case.

Key words: individual scaling, configuration, weights, quadratic programming, STRAIN, eigenvector, eigenvalue.

Introduction

In this note, we discuss the typical problem in individual scaling, namely, finding a common configuration and weights attached to dimensions for each individual from the given interpoint distances or scalar products. Tucker and Messick (1963), Horan (1969) and others have developed procedures for solving the problem. Carroll and Chang (1970) defined a minimization criterion (STRAIN) in terms of product moments computed from raw data. They use an alternative least square (ALS) method for estimating the configuration and weights. Within the STRAIN framework, Schönemann (1972) presented an algebraic solution in the case of exact data. Takane, Young and de Leeuw (1977) proposed a procedure called, ALSCAL in which the criterion function (SSTRESS) is in terms of distances obtained from raw data. The configuration and weights are obtained by solving certain normal equations in the least squares method alternately.

In this note, we consider the problem within the STRAIN framework and show that the problem of determining weights \((W_i)\) for a given configuration \((X)\) is posed as a standard quadratic programming problem for which efficient finitely convergent algorithms are available. A closed form expression for the configuration \((X)\) is obtained for the one dimensional case and an approach to determine \(X\) is given for the two dimensional case.

Problem Formulation

Given \(P_i: n \times n, (i = 1, \ldots, N)\) symmetric and positive semi-definite matrices for \(N\) individuals, the mathematical problem is to find a \(n \times t\) matrix \(X\) and \(t \times t\) diagonal matrices,

\[ W_i = \text{diag} (w_{i1}, \ldots, w_{ii}), \quad i = 1, \ldots, N, \]

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such that

\[ \text{STRAIN} = \sum_{i=1}^{N} \text{tr} \left( P_i - X W_i X' \right)^2 \]  

(1)
is minimum subject to the condition

\[ \frac{1}{N} \sum_{i=1}^{N} W_i = I_t, \quad \text{and} \]

(2)

\[ w_{ir} \geq 0, \quad i = 1, \ldots, N, \quad r = 1, \ldots, t. \]

\( I_t \) is the identity matrix of order \( t \) and \( \text{tr}(A) \) denotes the trace of matrix \( A \). When the optimal value of \( \text{STRAIN} \) is zero, we have

\[ P_i = X W_i X', \quad i = 1, \ldots, N. \]  

(3)

This situation is termed as the "exact case" and in this case, \( P_i \) is exactly decomposed into \( X W_i X' \). When the optimal value of the objective function in the problem given by (1) and (2) is strictly positive, it is called the "fallible case" and here it is not possible to find exactly \( X \) and \( W_i \) such that (3) holds.

Determination of \( W_i \) given \( X \): Quadratic Programming Formulation

For any given \( X \), the \( \text{STRAIN} \) is a function of \( W_i \) only and (1) can be written as

\[ \text{STRAIN} = \sum_{i=1}^{N} \text{tr}(X W_i X')^2 - 2 \sum_{i=1}^{N} \text{tr}(P_i X W_i X') + \sum_{i=1}^{N} \text{tr}(P_i^2). \]  

(4)

From (4), it can be seen that the objective function (1) is quadratic and convex in \( w_{ir} \), \( i = 1, \ldots, N, \quad r = 1, \ldots, t \). Note that \( \sum_{i=1}^{N} \text{tr}(P_i^2) \) is a given quantity which does not depend upon \( w_{ir} \). Thus for a given \( X \), the determination of optimal \( W_i = \text{diag} (w_{i1}, \ldots, w_{it}), \quad i = 1, \ldots, N \) is a convex quadratic programming problem with linear constraints (2). To solve this problem, efficient finitely convergent algorithms are available. See for example, Wolfe (1959), Cottle and Dantzig (1968) and Ravindran (1972). Thus exact optimum \( w_{ir} \) is obtained directly for a given \( X \).

Determination of \( X \) given \( W_i \)

Let \( X \) be an optimum solution to (1) for given \( W_i \) satisfying (2). Then a necessary condition satisfied by optimal \( X \) is clearly \( (\partial/\partial X)(\text{STRAIN}) = 0 \) or from (4),

\[ \sum_{i=1}^{N} (X W_i X' - P_i) X W_i = 0 \]  

(5)

Let \( y_r (r = 1, \ldots, t) \) be a column vector of \( X \). Then \( X = (y_1, \ldots, y_t) \). With this notation, (5) can be written as

\[ \begin{array}{ccc}
z_{11} y_1 y'_1 y_1 + & \cdots + & z_{1r} y_r y'_r y_1 = Q_1 y_1 \\
& \cdots & \\
& & \\
z_{rt} y_1 y'_1 y_t + & \cdots + & z_{tt} y_t y'_t y_t = Q_t y_t,
\end{array} \]  

(6)

where \( z_{rs}, s = 1, \ldots, t \) are the diagonal elements of

\[ \Delta_r = \sum_{i=1}^{N} W_i w_{ir}, \quad r = 1, \ldots, t; \]