On Bending of a Convex Surface to a Convex Surface with Prescribed Spherical Image

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ABSTRACT. We prove the following theorem. Let $F$ be a regular convex surface homeomorphic to the disk. Suppose the Gaussian curvature of $F$ is positive and the geodesic curvature of its boundary is positive as well. Let $G$ be a convex domain on the unit sphere bounded by a smooth curve and strictly contained in a hemisphere. Let $P$ be an arbitrary point on the boundary of $F$ and $P^*$ be an arbitrary point on the boundary of $G$. If the area of $G$ is equal to the integral curvature of the surface $F$, then there exists a continuous bending of the surface $F$ to a convex surface $F'$ such that the spherical image of $F'$ coincides with $G$ and $P^*$ is the image of the point in $F'$ corresponding to the point $P \in F$ under the isometry.

Theorem. Let $F$ be a regular convex surface homeomorphic to the disk. Suppose the Gaussian curvature of $F$ is positive and the geodesic curvature of its boundary is positive as well. Let $G$ be a convex domain on the unit sphere bounded by a smooth curve and contained strictly in a hemisphere. Finally let $P$ be an arbitrary point on the boundary of $F$ and $P^*$ be an arbitrary point on the boundary of $G$. If the area of $G$ is equal to the integral curvature of the surface $F$, then there exists a continuous bending of the surface $F$ to a convex surface $F'$ such that the spherical image of $F'$ coincides with $G$ and $P^*$ is the image of the point in $F'$ corresponding to the point $P \in F$ under the isometry.

Proof. Let us complete the surface $F$ regarded as a Riemannian manifold to a complete Riemannian manifold with nonnegative curvature. Then let us introduce semigeodesic coordinates $u, v$ in a neighborhood of the boundary of the surface $F$. For the family $v = \text{const}$, we take the geodesics orthogonal to the boundary, and for $u = \text{const}$ we take their orthogonal trajectories. We take the distance of a point from the boundary with the sign minus for $u$ and the distance along the boundary for $v$. In these coordinates the linear element of the surface $F$ is of the form

$$ds^2 = du^2 + G dv^2.$$  

where $G(0, v) = 1$. The Gaussian curvature is $K = -(\sqrt{G})_{uu}/\sqrt{G}$, and the geodesic curvature of the boundary is $\kappa = -(\sqrt{G})_u/\sqrt{G}$.

Now let us extend the metric on $F$ to a metric of the complete Riemannian manifold using the linear element (1). In order to do this, choose for $u \geq 0$ a smooth function $G$ satisfying the following condition: for $0 \leq u \leq \varepsilon$ ($\varepsilon$ is a small positive number)

$$\sqrt{G}(u, v) = 1 - \kappa(0, v)u - K(0, v)\left(\frac{u^2}{2} - \frac{u^4}{12\varepsilon^2}\right).$$

and $\sqrt{G}$ is a linear function over $u$ for $u > \varepsilon$. Denote such a complete manifold by $M_\varepsilon$. This manifold is supplied with a regular twice differentiable metric. The Gaussian curvature of $M_\varepsilon$ is nonnegative and it vanishes for $u > \varepsilon$. The total integral curvature of the manifold $M_\varepsilon$ is arbitrarily close to the total integral curvature of the surface $F$ for $\varepsilon$ small enough. And the total curvature of the surface $F$ is less than $2\pi$. Indeed, by the Gauss–Bonnet theorem,

$$\int_F K dS + \int_{\partial F} \kappa dv = 2\pi.$$
Since $x > 0$, we have $\int KdS < 2\pi$. This implies that for $\varepsilon$ small enough the integral curvature of the manifold $M_\varepsilon$ is less than $2\pi$.

According to a theorem of S. P. Olovyanishnikov [1, p. 429], the manifold $M_\varepsilon$ is isometric to an infinite complete convex surface. Denote this surface by $F_\varepsilon$. Taking the limit over an admissible sequence of surfaces $F_\varepsilon$ as $\varepsilon \to 0$, we obtain an infinite complete surface $F_0$. The surface $F_0$ has a bounded specific curvature in the following sense: for any domain on $F_0$ the ratio of the area of its spherical image to the area of the domain itself is bounded by a constant independent of the domain. The surface $F_0$ is smooth. Indeed, the surface $F_0$ cannot have conic points, since its specific curvature is bounded. It also cannot have edges. According to a theorem of A. D. Aleksandrov, an edge point on a surface with bounded specific curvature is an internal point of a rectilinear edge [2], p. 211. It then follows that the surface $F_0$ contains a line. As is well known, this surface is then cylindrical. And $F_0$ cannot be cylindrical since it contains a positive curvature domain. The surface $F_0$ consists of two parts: the part isometric to the surface $F$ (we denote this part by $F_0''$), and the complement, which we denote by $F_0'$. The surface $F_0''$ is locally isometric to the plane; its Gaussian curvature vanishes. Denote the boundary of the surface $F_0''$ by $\gamma$. Let us show that for any point of $\gamma$ there exists a linear ray starting in this point and lying on $F_0''$.

Let $A \in \gamma$ be an arbitrary point and let $A' \in F_0''$ be a point close to $A$. Consider the tangent plane at the point $A'$. This plane intersects the surface $F_0$ along some convex set $m$. This set contains points different from $A'$. Otherwise a small shift of the tangent plane would cut off a cap of positive curvature from the surface $F_0''$.

Let $\alpha$ be the plane containing the set $m$. The set $m$ cannot contain more than one point of strict convexity, i.e., a point such that there exists a line in $\alpha$ intersecting $m$ only by this point. Indeed, the set $m$ cannot have more than one point of intersection with the curve $\gamma$ since $\gamma$ has positive geodesic curvature. Therefore, if there is more than one point of strict convexity in $m$, then at least one of these points is an internal point of $F_0''$. Denote this point by $A''$. There exists a line $g$ in $\alpha$ intersecting $m$ only by $A''$. Shift the line $g$ in the plane $\alpha$ by a small vector directed to $m$ and rotate $\alpha$ by a small angle. The new plane will cut off a small cap from $F_0$. If the shift of $g$ and the angle of rotation are small enough, then the cap belongs to the surface $F_0''$. This is impossible, however, since $F_0$ has zero curvature. Thus, $m$ cannot have more than one point of strict convexity. The set $m$ is then either a half-line, or a plane angle, or a half-plane. It cannot be a half-plane, since otherwise $F_0$ would contain a line. Making the point $A'$ tend to the point $A$, we conclude that there exists a half-line in the tangent plane at $A$ lying on the surface $F_0''$.

Suppose now that the spherical image of the surface $F_0$ is a convex domain on the unit sphere bounded by a smooth curve $\gamma^*$. Let us show that the ray starting in $A$ and lying on $F_0''$ is unique. Denote the spherical image of $A$ by $A^*$, $A^* \in \gamma^*$. If there are different rays, then all the rays form some angle in the tangent plane to the surface $F_0$. Let $B$ be a point on a side of this angle and let $B' \in F_0''$ be a point close to $B$ and outside of this angle. Denote their spherical images by $B^*$ and $B'^*$. The points $B^*$ and $A^*$ coincide. The intersection of the tangent planes to $F_0$ at the points $B$ and $B'$ and the intersection of the tangent planes to the sphere at the points $A^*$ and $B'^*$ are parallel lines. Let the point $B'$ tend to the point $B$. The intersection line of the tangent planes in the points $B$ and $B'$ tends to the line containing the side of the angle. The intersection line of the tangent planes at the points $A^*$ and $B'^*$ tends to the line orthogonal to $\gamma^*$ in the point $A^*$. This implies that the ray starting from $A$ is unique, since there exists a unique line tangent to the sphere at the point $A^*$ and orthogonal to $\gamma$.

There are a lot of degrees of freedom in bending (isometrically transforming) an infinite complete convex surface with curvature less than $2\pi$. In order to characterize this freedom, let us recall the definition of a limit cone of a surface and of a geodesic ray on a surface. The limit cone of a surface is the cone obtained as the limit of the homothetic contraction of the surface. The spherical image of the limit cone coincides with that of the surface itself. A geodesic ray is an infinite geodesic unbounded in one direction such that any finite segment of it is the shortest path. The direction of a tangent half-line to a geodesic ray tends to one of the elements of the limit cone as the point of tangency tends to infinity. This limit element is called the limit element of the geodesic ray.

The Olovyanishnikov theorem we referred to above claims the following. Let $M$ be a complete manifold with nonnegative curvature and with total curvature less than $2\pi$. Suppose $\gamma$ is a geodesic ray in $M$,