A Test for Compactness of a Foliation

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ABSTRACT. We investigate foliations on smooth manifolds that are determined by a closed 1-form with Morse singularities. We introduce the notion of the degree of compactness and prove a test for compactness.

In the present paper we investigate foliations on smooth manifolds that are determined by a closed 1-form with Morse singularities. The problem of investigating the topological structure of level surfaces for such a form was posed by S. P. Novikov in [1]. This problem was treated in [2-5]. The present paper is devoted to the compactness problem for level surfaces. We introduce the notion of degree of compactness and prove a test for compactness expressed in the terms of the degree.

In §1 we give the necessary definitions and define the degree of compactness. The central result of the paper, i.e., the test for compactness of a foliation, is proved in §2. In §3 we present some consequences: a relation between the degree of compactness and the degree of irrationality of the form, and a more detailed investigation of the two-dimensional case.

The present paper is a natural continuation of [6].

§1. Preliminary definitions

Consider a smooth compact n-dimensional manifold M and a closed 1-form \( \omega \) on M with nondegenerate isolated singularities.

**Definition** 1 [7]. A point \( p \in M \) is said to be a regular singularity of the differential form \( \omega \) if in a neighborhood \( O(p) \) we have \( \omega = df \), where \( f \) is a Morse function with a singularity at \( p \). There exist, therefore, coordinates \( x^1, \ldots, x^n \) such that in this neighborhood we have

\[
\omega = \sum_{i=1}^k x_i dx^i - \sum_{i=k+1}^n x_i dx^i.
\]

The number \( \min(k, n-k) \) is called the index of the singular point.

On the set \( M - \text{Sing}\omega \) the form \( \omega \) determines a foliation \( \mathcal{F}_\omega \) of codimension 1. If the index of the singular point \( P \) is equal to zero, then there exists a foliation of a neighborhood of \( P \) into spheres. If \( \text{ind} P = 1 \), then there exists a fiber that becomes locally arcwise connected after adding the singular point \( P \). This fiber is called the canonical fiber. For \( \text{ind} P > 1 \) all the fibers in the neighborhood of \( P \) are locally arcwise connected.

The foliation \( \mathcal{F}_\omega \) contains fibers of three kinds [7]:

1) compact fibers admitting a neighborhood consisting of diffeomorphic fibers;
2) conic fibers, i.e., fibers that may be made locally arcwise connected in a neighborhood of a singular point by adding this singular point to the fiber;
3) all the other noncompact fibers.

Below we assume that the singular point \( P \) belongs to the fiber, and thus all the fibers are arcwise connected.
Definition 2. Consider a compact fiber $\gamma$ of $\mathcal{F}_\omega$ and the mapping $\gamma \mapsto [\gamma] \in H_{n-1}(M)$. Then the images of all compact fibers are spanned by a subgroup in $H_{n-1}(M)$. We denote this subgroup by $H_\omega$ and call $\text{rk} H_\omega$ the degree of compactness of the foliation $\mathcal{F}_\omega$.

Since $H_\omega \subseteq H_{n-1}(M)$, we obtain $0 \leq \text{rk} H_\omega \leq \beta_{n-1}$, where $\beta_{n-1} = \text{rk} H_{n-1}(M)$. If all fibers of $\mathcal{F}_\omega$ are noncompact, then $\text{rk} H_\omega = 0$. The converse is false: some compact fibers may prove to be homologous to zero. Moreover, there exist compact foliations with $\text{rk} H_\omega = 0$. To obtain such a foliation, it is sufficient to consider a manifold $M$ such that $H_{n-1}(M) = 0$. The foliation associated with any closed form is obviously compact, and all the fibers are homologous to zero.

Consider the group $H_{n-1}(M)$ and the intersection map

$$o : H_{n-1}(M) \times H_{n-1}(M) \to H_{n-2}(M)$$

for homology classes, which is defined in the following way [8]. Let $x, y \in H_{n-1}(M)$, and let $D$ denote Poincaré duality; then $x \circ y = D x \cap y$. If the homology classes $x$ and $y$ are realized by the submanifolds $X$ and $Y$, then $x \circ y$ is the homology class of $X \cap Y$. The intersection is skew-symmetric: $x \circ y = - y \circ x$.

Definition 3. Consider the subgroup $H \subset H_{n-1}(M)$ such that for all $x, y \in H$ we have $x \circ y = 0$. We call $H$ the isotropic subgroup with respect to the intersection of cycles. An isotropic subgroup $H$ is called maximal, if for all $x \in H$, $x \neq 0$ and $y \notin H$ we have $x \circ y \neq 0$.

The subgroup $H_\omega$ of compact fibers obviously is an isotropic subgroup in $H_{n-1}(M)$.

Denote by $M_\omega$ the set obtained by eliminating from $M$ all the maximal neighborhoods consisting of diffeomorphic compact fibers and all the fibers that may be compactified by adding singular points.

§2. The main theorem

Let us establish the validity of the following test.

Theorem. If the subgroup $H_\omega$ spanning all the compact fibers is a maximal isotropic subgroup of the homology group $H_{n-1}(M)$, then $M_\omega = \emptyset$.

Proof. Suppose the subgroup $H_\omega$ has maximal rank, and let $H_\omega = \langle [\gamma_1], \ldots, [\gamma_N] \rangle$, where $\gamma_i$ are fibers of $\mathcal{F}_\omega$. Cutting $M$ along the fibers $\gamma_1, \ldots, \gamma_N$, we obtain a manifold $M'$ with boundary.

Let $\varphi : M' \to M$ be the gluing map, and let $i : \partial M' \to M'$ be the boundary inclusion mapping.

Lemma 1. If $H_\omega$ is a maximal isotropic subgroup, then the mapping $i_* : H_{n-1}(\partial M') \to H_{n-1}(M')$ is surjective.

Proof. The gluing map $\varphi : M' \to M$ induces the mapping of pairs $\varphi : (M', \partial M') \to (M, \bigcup \gamma_i)$. Let us set $\varphi|_{\partial M'} = \varphi_1$ and consider the commutative diagram

$$H_{n-1}(\partial M') \xrightarrow{i_*} H_{n-1}(M')$$

$$\downarrow \varphi_1 \downarrow \varphi$$

$$H_{n-1}(\bigcup \gamma_i) \xrightarrow{j} H_{n-1}(M).$$

We claim that 1) the mapping $j$ is injective, 2) the mapping $\varphi_1$ is surjective, 3) $\ker \varphi_1 \subseteq \text{im} i_*$.\[1)\] Since the fibers $\gamma_i$ do not intersect, $\gamma_i \cap \gamma_j = \emptyset$, the Mayer–Vietoris exact sequence gives

$$H_{n-1}(\bigcup \gamma_i) = \oplus H_{n-1}(\gamma_i).$$

By assumption, the cycles $[\gamma_i]$ are independent in $M$, and, therefore,

$$\oplus H_{n-1}(\gamma_i) = \langle [\gamma_1], \ldots, [\gamma_N] \rangle = H_\omega$$

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