Some Topics on the Continuation and Smoothing of Vector Functions
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Abstract. We consider problems of continuation of vector functions from a subspace to the entire space and of smoothing problems for these functions. It is shown that there exists a reflexive separable space \( X \) and a subspace \( Y \) such that even a very smooth mapping of \( Y \) does not extend to a uniformly continuous mapping of a neighborhood of \( Y \).

Let \((B)\) be the class of all real Banach spaces. For any \( X \in (B) \) by \( \omega_X(\varepsilon) \) \((\varepsilon > 0)\) we denote the modulus of convexity of \( X \), that is,

\[
\omega_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.
\]

Let \((UR)\) be the class of uniformly convex spaces \( X \in (B) \), that is, Banach spaces \( X \) such that \( \omega_X(\varepsilon) > 0 \) for all \( \varepsilon > 0 \). By \((SRf)\) we denote the class of all superreflexive spaces \( X \in (B) \), that is, spaces that can be made uniformly convex by proceeding to an equivalent norm. Let \( X \in B \); then \( X^* \) is the dual space of \( X \), and \( X^{**} \) is the dual space of \( X^* \). Let \((RF)\) be the class of all reflexive spaces \( X \in (B) \), that is, spaces such that \( X = X^{**} \). By \( B(x, r) = \{y \in X : \|y - x\| < r\} \) we denote the ball of radius \( r \geq 0 \) centered at \( x \).

Suppose that \( X, Y \in (B), M \subseteq X \) is an arbitrary nonempty convex set, and \( f: M \to Y \). Let \( k \in \mathbb{N} \), \( x \in M \), and \( h \in X \), and suppose that \([x, x + kh] \subseteq M\); then the \( k \)th difference of \( f \) at \( x \) with increment \( h \) is defined as the expression

\[
\Delta_k^h(f, x) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i f(x + ih).
\]

We define the \( k \)th-order modulus of continuity of \( f \) as the function \( \omega_k(f, u) \) defined on \( \mathbb{R}_+ \) by the formula

\[
\omega_k(f, u) = \sup \{ \|\Delta_k^h(f, x)\|_Y : h \in X, \|h\| \leq u, [x, x + kh] \subseteq M \}.
\]

For \( k = 1 \), instead of \( \omega_1(f, u) \), we write \( \omega(f, u) \). Let \( M \subset X \) be an arbitrary nonempty set. Then by \( C(M, Y) \) we denote the space of all continuous mappings \( f: M \to Y \) equipped with the norm

\[
\|f\| = \sup_{x \in M} \|f(x)\|_Y < \infty \quad (f \in C(M, Y)).
\]

Let \( \Phi^k \) \((k \in \mathbb{N})\) be the set of all monotone continuous functions \( \omega: [0, \infty) \to \mathbb{R}_+ \) such that \( \omega(0) = 0 \) and \( \omega(nt) \leq n^k \omega(t) \) for any \( t \in [0, \infty) \) and \( n \in \mathbb{N} \). For an arbitrary \( \omega \in \Phi^1 \), by \( H^\omega(M, Y) \) we denote the set of all \( f \in C(M, Y) \) such that \( \omega(f, u) \leq \omega(u) \) \((u \in \mathbb{R}_+)\). Let \( CLip(M, Y) \) be the set of all \( C \)-Lipschitz mappings \( f: M \to Y \), that is, mappings such that \( \|f(x) - f(y)\| \leq C\|x - y\| \) for any \( x, y \in M \).

By \( UC(M, Y) \) we denote the set of all uniformly continuous mappings \( f: M \to Y \). If \( M \subset X \) is a body, then by \( UC^r(M, Y) \) \((r \in \mathbb{N})\) we denote the set of all mappings \( f: M \to Y \) whose \( r \)th Fréchet derivative exists and is uniformly continuous on \( M \).

For any \( \omega \in \Phi^1 \), an arbitrary body \( M \subset X \in (B) \), and any \( Y \in (B) \), consider the set

\[
K_\omega(M, Y) = \{ f: M \to Y : \omega(f', t) \leq \omega(t), \ t > 0 \}.
\]

Finally, set

\[
E(f, K) = \inf_{\varphi \in K} \|f - \varphi\| \quad (f \in C(M, Y), \ K \subset C(M, Y)).
\]
**Theorem 1.** Suppose that \( X \in (B), Y \in (SRf), \dim X = \infty, \text{ and } \dim Y = \infty \) and there exist \( f \in UC(B, Y) \) \((B \subset X \text{ is the unit ball})\) such that \( E(f, UC(B, Y)) > 0 \).

**Proof.** We equip \( Y \) with some norm with respect to which \( Y \) belongs to \((UR)\). It was proved in [1] that for any \( Z \in (B) \) with \( \dim Z = \infty \) and any \( n \in \mathbb{N} \) there exists a subspace \( L \subset Z \) with \( \dim L = n \) and a Euclidean norm \( \| \cdot \|_1 \) such that

\[
\frac{1}{2} \| y \| \leq \| y \|_1 \leq 2 \| y \| \quad \text{for all } y \in L.
\]

Let \( X_n \subset X \) and \( Y_n^* \subset Y^* \) be \( n \)-dimensional subspaces such that there exist Euclidean norms \( \| \cdot \|_i \) \((i = 1, 2)\) satisfying

\[
\frac{1}{2} \| y_i \| \leq \| y_i \|_i \leq 2 \| y_i \| \quad (i = 1, 2)
\]

for all \( y_1 \in X_n \) and \( y_2 \in Y_n^* \). Consider the subspace \( L_n = \{ y \in Y \mid \forall y^* \in Y_n^* \ y^*(y) = 0 \} \) and the quotient space \( V_n = Y/L_n \). Then there exists a Euclidean norm \( \| \cdot \|_3 \) on \( V_n \) such that

\[
\frac{1}{2} \| v \| \leq \| v \|_3 \leq 2 \| v \| \quad \text{for all } v \in V_n.
\]

It is in fact proved in [2] that for a separable Hilbert space \( H \) and the unit ball \( B_0 \subset H \) there exists an \( I \)-Lipschitz function \( \varphi: H \rightarrow H \) that cannot be approximated by the class \( UC^1(B_0, H) \) on \( B_0 \), that is, \( \Delta = E(\varphi|B_0, UC^1(B_0, H)) > 0 \). Let \( \{ H_m \} \) be an arbitrary sequence of Euclidean spaces of dimension \( m \in \mathbb{N} \), and let \( B_m \subset H_m \) be the unit ball \((m \in \mathbb{N})\). It follows that there exists a sequence of \( I \)-Lipschitz functions \( \varphi_m: H_m \rightarrow H \) such that

\[
\lim_{m \to \infty} E(\varphi_m|B_m, K_\omega(B_m, H_m)) > 0
\]

for all \( \omega \in \Phi^1 \). Indeed, let us assume that \( H_m \subset H_{m+1} \subset H \) \((m \in \mathbb{N})\) and that \( \mathcal{L} = \bigcup_{m=1}^\infty H_m \) is everywhere dense in \( H \). Let \( P_m: H \rightarrow H_m \) be the orthogonal projection on \( H_m \) \((m \in \mathbb{N})\). Set \( \varphi_m = P_m \circ \varphi \circ P_m \). It is easy to see that \( \{ \varphi_m \} \) is a sequence of \( I \)-Lipschitz functions that is pointwise convergent to \( \varphi \) on \( H \).

**Lemma 1.** For any \( \omega \in \Phi^1 \) we have the inequality

\[
\lim_{n \to \infty} E(\varphi_n|B_n, K_\omega(B_n, H)) > \frac{\Delta}{2}.
\]

**Proof.** Assume the converse. Then there exists an \( \omega \in \Phi^1 \) and a sequence of functions \( \psi_m \in K_\omega(B_m, H_m) \) such that \( \| \varphi_m|B_m - \psi_m \| \leq \Delta/2 \). Since the sequence \( \{ g_m = \psi_m \circ P_m \} \subset K_\omega(B_0, H) \) is equicontinuous and bounded, it follows that there exists a subsequence \( \{ g_{m_k} \} \) uniformly convergent on each \( B_n \) \((n \in \mathbb{N})\) (on the image we consider the metric that generates the weak topology on the \( \Delta \)-neighborhood of \( \varphi(B_0) \)). It follows that \( \{ g_{m_k} \} \) is pointwise convergent on \( B_0 \) to some \( g \in K_\omega(B_0, H) \) (we equip the image with the weak topology). Since

\[
\| g_{m_k}(x) - \varphi_{m_k}(x) \| \leq \frac{\Delta}{2} \quad \text{for all } x \in \bigcup_{m=1}^\infty B_m,
\]

it follows that \( \| g(x) - \varphi(x) \| \leq \Delta/2 \). Consequently, \( \| g - \varphi|B_0 \| \leq \Delta/2 \), which is a contradiction. \( \square \)

It follows from Lemma 1 that there exists a sequence of \( I \)-Lipschitz functions \( \psi_m: X_m \rightarrow Y_m^* \) \((m \in \mathbb{N})\) such that

\[
\lim_{m \to \infty} E(\psi_m|D_m, K_\omega(D_m, Y_m^*)) > \frac{\Delta}{2}
\]