Some Topics on the Continuation and Smoothing of Vector Functions

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Abstract. We consider problems of continuation of vector functions from a subspace to the entire space and of smoothing problems for these functions. It is shown that there exists a reflexive separable space X and a subspace Y such that even a very smooth mapping of Y does not extend to a uniformly continuous mapping of a neighborhood of Y.

Let (B) be the class of all real Banach spaces. For any $X \in (B)$ by $\omega_X(\varepsilon)$ ($\varepsilon > 0$) we denote the modulus of convexity of $X$, that is,

$$
\omega_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| \mid \left\| x \right\| = \left\| y \right\| = 1, \left\| x - y \right\| \geq \varepsilon \right\}.
$$

Let (UR) be the class of uniformly convex spaces $X \in (B)$, that is, Banach spaces $X$ such that $\omega_X(\varepsilon) > 0$ for all $\varepsilon > 0$. By (SRf) we denote the class of all superreflexive spaces $X \in (B)$, that is, spaces that can be made uniformly convex by proceeding to an equivalent norm. Let $X \in B$; then $X^*$ is the dual space of $X$, and $X^{**}$ is the dual space of $X^*$. Let (RF) be the class of all reflexive spaces $X \in (B)$, that is, spaces such that $X = X^{**}$. By $B(x, r) = \{ y \in X \mid \left\| y - x \right\| < r \}$ we denote the ball of radius $r \geq 0$ centered at $x$.

Suppose that $X, Y \in (B)$, $M \subset X$ is an arbitrary nonempty convex set, and $f: M \to Y$. Let $k \in \mathbb{N}$, $x \in M$, and $h \in X$, and suppose that $[x, x + kh] \subset M$; then the $k$th difference of $f$ at $x$ with increment $h$ is defined as the expression

$$
\Delta_k^h(f, x) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i f(x + ih).
$$

We define the $k$th-order modulus of continuity of $f$ as the function $\omega_k(f, u)$ defined on $\mathbb{R}^+$ by the formula

$$
\omega_k(f, u) = \sup \{ \left\| \Delta_k^h(f, x) \right\|_Y \mid h \in X, \left\| h \right\| \leq u, [x, x + kh] \subset M \}.
$$

For $k = 1$, instead of $\omega_1(f, u)$, we write $\omega(f, u)$. Let $M \subset X$ be an arbitrary nonempty set. Then by $C(M, Y)$ we denote the space of all continuous mappings $f: M \to Y$ equipped with the norm

$$
\left\| f \right\| = \sup_{x \in M} \left\| f(x) \right\|_Y < \infty \quad (f \in C(M, Y)).
$$

Let $\Phi^k$ ($k \in \mathbb{N}$) be the set of all monotone continuous functions $\omega: [0, \infty) \to \mathbb{R}^+$ such that $\omega(0) = 0$ and $\omega(nt) \leq n^k \omega(t)$ for any $t \in [0, \infty)$ and $n \in \mathbb{N}$. For an arbitrary $\omega \in \Phi^1$, by $H^\omega(M, Y)$ we denote the set of all $f \in C(M, Y)$ such that $\omega(f, u) \leq \omega(u)$ ($u \in \mathbb{R}^+$). Let $C\text{Lip}(M, Y)$ be the set of all $C$-Lipschitz mappings $f: M \to Y$, that is, mappings such that $\left\| f(x) - f(y) \right\| \leq C\|x - y\|$ for any $x, y \in M$.

By $UC(M, Y)$ we denote the set of all uniformly continuous mappings $f: M \to Y$. If $M \subset X$ is a body, then by $UC^r(M, Y)$ ($r \in \mathbb{N}$) we denote the set of all mappings $f: M \to Y$ whose $r$th Fréchet derivative exists and is uniformly continuous on $M$.

For any $\omega \in \Phi^1$, an arbitrary body $M \subset X \in (B)$, and any $Y \in (B)$, consider the set

$$
K_{\omega}(M, Y) = \{ f: M \to Y \mid \omega(f', t) \leq \omega(t), \ t > 0 \}.
$$

Finally, set

$$
E(f, K) = \inf_{\varphi \in K} \| f - \varphi \| \quad (f \in C(M, Y), K \subset C(M, Y)).
$$
Theorem 1. Suppose that \( X \in (B), \ Y \in (SRf) \), \( \dim X = \infty \), and \( \dim Y = \infty \). Then there exists a \( f \in UC(B, Y) \) \((B \subset X \) is the unit ball\) such that \( E(f, UC(B, Y)) > 0 \).

Proof. We equip \( Y \) with some norm with respect to which \( Y \) belongs to (UR). It was proved in [1] that for any \( Z \in (B) \) with \( \dim Z = \infty \) and any \( n \in \mathbb{N} \) there exists a subspace \( L \subset Z \) with \( \dim L = n \) and a Euclidean norm \( \| \cdot \|_1 \) such that

\[
\frac{1}{2} \| y \| \leq \| y \|_1 \leq 2 \| y \| \quad \text{for all} \quad y \in L.
\]

Let \( X_n \subset X \) and \( Y_n^* \subset Y^* \) be \( n \)-dimensional subspaces such that there exist Euclidean norms \( \| \cdot \|_1 \) (\( i = 1, 2 \)) satisfying

\[
\frac{1}{2} \| y_i \| \leq \| y_i \|_1 \leq 2 \| y_i \| \quad (i = 1, 2)
\]

for all \( y_1 \in X_n \) and \( y_2 \in Y_n^* \). Consider the subspace \( L_n = \{ y \in Y \mid \forall y^* \in Y_n^* \ y^*(y) = 0 \} \) and the quotient space \( V_n = Y/L_n \). Then there exists a Euclidean norm \( \| \cdot \|_3 \) on \( V_n \) such that

\[
\frac{1}{2} \| v \| \leq \| v \|_3 \leq 2 \| v \| \quad \text{for all} \quad v \in V_n.
\]

It is in fact proved in [2] that for a separable Hilbert space \( H \) and the unit ball \( B_0 \subset H \) there exists an \( I \)-Lipschitz function \( \varphi : H \to H \) that cannot be approximated by the class \( UC^1(0, H) \) on \( B_0 \), that is, \( \Delta = E(\varphi|B_0, UC^1(B_0, H)) > 0 \). Let \( \{ H_m \} \) be an arbitrary sequence of Euclidean spaces of dimension \( m \in \mathbb{N} \), and let \( B_m \subset H_m \) be the unit ball \((m \in \mathbb{N})\). It follows that there exists a sequence of \( I \)-Lipschitz functions \( \varphi_m : H_m \to H_m \) such that

\[
\lim_{m \to \infty} E(\varphi_m|B_m, K_n(B_m, H_m)) > 0
\]

for all \( \omega \in \Phi^1 \). Indeed, let us assume that \( H_m \subset H_{m+1} \subset H \) \((m \in \mathbb{N})\) and that \( L = \bigcup_{m=1}^{\infty} H_m \) is everywhere dense in \( H \). Let \( P_m : H \to H_m \) be the orthogonal projection on \( H_m \) \((m \in \mathbb{N})\). Set \( \varphi_m = P_m \circ \varphi \circ P_m \). It is easy to see that \( \{ \varphi_m \} \) is a sequence of \( I \)-Lipschitz functions that is pointwise convergent to \( \varphi \) on \( H \).

Lemma 1. For any \( \omega \in \Phi^1 \) we have the inequality

\[
\lim_{n \to \infty} E(\varphi_n|B_n, K_n(B_n, H)) > \frac{\Delta}{2}.
\]

Proof. Assume the converse. Then there exists an \( \omega \in \Phi^1 \) and a sequence of functions \( \psi_m \in K_n(B_m, H_m) \) such that \( \| \varphi_m - \psi_m \| \leq \Delta/2 \). Since the sequence \( \{ g_m = \psi_m \circ P_m \} \subset K_n(B_0, H) \) is equicontinuous and bounded, it follows that there exists a subsequence \( \{ g_{m_k} \} \) uniformly convergent on each \( B_n \) \((n \in \mathbb{N})\) (on the image we consider the metric that generates the weak topology on the \( \Delta \)-neighborhood of \( \varphi(B_0) \)). It follows that \( \{ g_{m_k} \} \) is pointwise convergent on \( B_0 \) to some \( g \in K_n(B_0, H) \) (we equip the image with the weak topology). Since

\[
\| g_{m_k}(x) - \varphi_{m_k}(x) \| \leq \frac{\Delta}{2} \quad \text{for all} \quad x \in \bigcup_{m=1}^{\infty} B_m,
\]

it follows that \( \| g(x) - \varphi(x) \| \leq \Delta/2 \). Consequently, \( \| g - \varphi \|_{B_0} \leq \Delta/2 \), which is a contradiction. \( \square \)

It follows from Lemma 1 that there exists a sequence of \( I \)-Lipschitz functions \( \psi_m : X_m \to Y_m^* \) \((m \in \mathbb{N})\) such that

\[
\lim_{m \to \infty} E(\psi_m|D_m, K_n(D_m, Y_m^*)) > \frac{\Delta}{2}
\]