Critical Sets and Unimodal Mappings of the Square

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ABSTRACT. A lower estimate for the number of different invariant subsets of the set of nonwandering points for a class of unimodal mappings is given. Sufficient conditions for such a mapping to have periodic points of arbitrarily large period are described. The machinery of the appearance of such points may be of very different nature. The existence of mappings with trajectory behavior chaotic in the Li-York sense is established. Conditions for the domain of these trajectories to be arbitrary small are given. Therefore, such trajectories cannot be found by numerical methods.

Investigation of the images of the critical line for a mapping allows one to isolate the domains with chaotic behavior of trajectories [1, 2]. The investigation of the inverse images of the critical sets together with the critical sets themselves yields an estimate of the number of different invariant sets for unimodal mappings of the square $I^2 = [0, 1] \times [0, 1]$. The existence of mappings with chaotic trajectory behavior in the Li-York sense [3] may then be shown analytically.

Let $\text{Int}(T)$ be the set of interior points of the set $T \subset I^2$, let $\Gamma(T) = T \setminus \text{Int}(T)$ be the boundary of $T$, and let $U_\varepsilon(T)$ be the $\varepsilon$-neighborhood of $T$ in $I^2$. Denote by $\|DG\|$ the determinant of the Jacobian $DG$ of the mapping $G: I^2 \rightarrow I^2$ at a point $(x, y)$; let $NW(G)$ be the set of nonwandering points and

$$G^n = G(G(\ldots(G(x, y))\ldots)),$$

$n$ times

Definition. A mapping $(x, y) \rightarrow (\theta(x, y), x)$ of the square $I^2$ into itself is called unimodal if $\theta(x, y)$ is a $C^1$-smooth function such that $\theta(x, 0) = \theta(x, 1) = 0$ and $\theta(0, y) = \theta(1, y) = 0$ and there exists a point $(\bar{x}, \bar{y})$ satisfying $\theta(\bar{x}, \bar{y}) = \theta(x, y) > 0$ for any $(x, y) \in \text{Int}(I^2)$, where $(\bar{x}, \bar{y})$ is the only point where the partial derivatives $\theta_x$ and $\theta_y$ vanish simultaneously.

Consider the following family of mappings:

$$F: (x, y) \rightarrow (\lambda f(x)[g(y)]^s, x), \tag{1}$$

where $0 < \lambda \leq 1$, $s > 0$ are real parameters, $f(t)$ is a $C^1$-smooth unimodal function on $[0, 1]$ such that $f(0) = f(1) = 0$, $f(q) = 1$, the point $q$ being the extremal point of $f(t)$, $(t - q)f'(t) < 0$ on $[0, 1] \setminus \{q\}$; here $f'(t)$ is the derivative of $f(t)$ and $g(t) = h[f(t)]$, where $h(t)$ is a $C^1$-diffeomorphism of $[0, 1]$ such that $h(0) = 0$ and $h'(t) > 0$ on $[0, 1]$.

The family $F$ is chosen in the form (1), because this specific form of the function $\theta(x, y)$ yields simple analytical expressions for the first two preimages and the first two images of the critical set. This allows to visualize their mutual configuration and simplifies the analysis.

Definition. The closure of the set $\{(x, y) \in \text{Int}(I^2) : \|DG\| = 0\}$ is said to be the critical set $K = K(G)$ of the mapping $G$.

The critical set does not coincide with the "critical line LC" [1], although the two are closely related: $LC \subset G(K)$. For $G = F$ we have $LC = F(K)$. Let us explicitly write $K(F)$ and its first two images and preimages.

$$K = \{(x, y) \in I^2 : y = q, \ 0 \leq x \leq 1\},$$

$$F(K) = \{(x, y) \in I^2 : x = \lambda f(y), \ 0 \leq y \leq 1\},$$
\[ F^{-1}(K) = \{(x, y) \in I^2 : x = q, 0 \leq y \leq 1 \}, \]
\[ F^2(K) = \{(x, y) \in I^2 : x = \lambda f(y)[h(y/\lambda)]^s, 0 \leq y \leq \lambda \}, \]
\[ F^{-2}(K) = \{(x, y) \in I^2 : q = \lambda f(x)[h(f(y))]^s \}. \]

Note that \( F^{-2}(K) \) exists for \( \lambda \geq q \).

Introduce the notation \( R = F^2(I^2), R^{(n)} = F^{n+2}(I^2) \). It is easy to show that
\[ R = \{(x, y) \in I^2 : 0 < y < \lambda \} \text{ for } h(t) = t. \]

The critical set \( K \) and the preimages \( F^{-j}(K), j = 1, \ldots, J \) separate \( R^{(n)} \) into compact subsets by means of the arcs \([R^{(n)} \cap F^{-j}(K)]_{j=0}^{J-1} \).

We suppose below that \( h(t) = t \).

**Definition.** A connected component of the boundary \( \Gamma(T) \) is called the base of \( T \) if it belongs to an arc from \([R^{(n)} \cap F^{-j}(K)]_{j=0}^{J-1} \). Bases belonging to different arcs are regarded as different. The complement of the bases to \( \Gamma(T) \) consists of the sides \( T \). Each connected component forms one side.

Let \( N = N(n, J) \) be the number of the connected components in \( R^{(n)} \), and let \( i \in \{1, 2, \ldots, N\} \) be the \( i \)th connected component of \( R^{(n)} \), \( i = 1, \ldots, N \) and \( L, M \in \mathbb{N} \).

**Definition.** The sets \( F^L(j) \) and \( i \) intersect regularly in dimension one if 1) \( i \) has only two bases, and these bases do not intersect each other or the \( F^L \)-images of the bases of \( j \); and 2) there exists a connected component of \( \xi \subset F^L(j) \cap i \) having only two bases belonging to different bases of \( i \), and the sides of \( \xi \) do not intersect the sides of \( i \).

**Definition.** The sets \( F^L(j) \) and \( i \) intersect regularly in dimension two if \( \text{Int}(F^L(j)) \supset i \).

According to either of the definitions of regularity, let us set the entry \( \pi_{ji} \) of the matrix \( \Pi \) equal to 1 if the sets \( F^L(j) \) and \( i \) intersect regularly and equal to zero otherwise. Now consider all sequences of the elements \( 1, \ldots, N \) infinite to the right such that the entry \( \pi_{ji} \) corresponding to a pair \( j, i \) of neighboring elements equals 1.

**Lemma 1.** Any periodic sequence of period \( M \) corresponds to at least one compact set \( \Omega \subset NW(F) \), invariant with respect to \( F^{LM} \), such that for any point \( \omega \in \Omega \) we have \( F^{LM} (\omega) = i_m \) for \( m = 1, 2, \ldots, \), where \( i_m = i(m) \) is the \( m \)th element in the sequence and \( i_M = i_1, i_{M+1} = i_2, \) etc., and so on.

**Proof.** The idea is to isolate the subset of \( NW(F) \) corresponding to the given sequence. We are arguing in the case of regularity in dimension one. In dimension two the argument is similar.

Denote by \( \xi_1 \) the connected components of the decomposition of \( R^{(n)} \), i.e., \( \xi_1 \in \{1, 2, \ldots, N\} \), and let the elements \( \{\xi_i\}, i = 1, 2, 3, \ldots \) follow each other, so that \( F^L(\xi_1) \) and \( \xi_{i+1} \) intersect regularly. Let \( \tilde{\xi}_1 \) be an arbitrary "zone" satisfying the regularity intersection conditions for the sets \( F^L(\xi_1) \) and \( \xi_2 \). Denote by \( F^{-L}(\tilde{\xi}_1) \) the preimage of \( \tilde{\xi}_1 \) in \( \xi_1 \). Since the sets \( F^L(\xi_2) \) and \( \xi_3 \) intersect regularly, the sets \( F^L(\tilde{\xi}_1) \) and \( \xi_3 \) intersect regularly as well. Therefore, there exists a "zone" \( \tilde{\xi}_2 \) in \( F^L(\tilde{\xi}_1) \cap \xi_3 \) satisfying the above conditions. Denote by \( F^{-2L}(\tilde{\xi}_2) \) the preimage of \( \tilde{\xi}_2 \) in \( \xi_1 \). Obviously, \( F^{-2L}(\tilde{\xi}_2) \subset F^{-L}(\tilde{\xi}_1) \subset \xi_1 \), and so on. The sets \( F^{-Lj}(\tilde{\xi}_j) \) form a sequence of nested closed sets and therefore have a nonempty intersection. Denote
\[ \Psi = \bigcap_{j=1}^{\infty} F^{-Lj}(\tilde{\xi}_j) \quad \text{and} \quad \Omega = \Psi \cap NW(F). \]

Obviously, for any \( \omega \in \Omega \) we have \( F^L(\omega) \in \xi_1, i \in N \).

Now let us show that \( \Omega \neq \emptyset \). \( M \)-periodicity implies \( \tilde{\xi}_M \subset \xi_1 \). On the other hand,
\[ F^{-LM}(\tilde{\xi}_M) \subset \xi_1, \quad \text{and} \quad \tilde{\xi}_M \cap F^{-LM}(\xi_M) \subset \xi_1. \]