A Construction of Complete Minimal, But Not Uniformly Minimal, Exponential Systems with Real Separable Spectrum in \( L^p \) and \( C \)

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ABSTRACT. We construct real separable sequences \( \{\lambda_n\} \) such that the corresponding systems of exponentials \( \exp(i\lambda_n t) \) are complete and minimal, but not uniformly minimal, in the spaces \( L^p(-\pi, \pi) \), \( 1 \leq p < \infty \), or \( C[-\pi, \pi] \).

§1. In the analysis of approximate properties of exponential systems

\[
\exp(i\lambda_n t), \quad \lambda_n \in \Lambda, \tag{1}
\]
on a finite interval (say, \([-\pi, \pi]\)), the researchers' attention, starting from [1], is chiefly attracted to such questions as completeness and minimality of systems (1) in \( L^p = L^p(-\pi, \pi), \ 1 \leq p < \infty \), and \( C = C[-\pi, \pi] \), unconditional bases of the form (1) in \( L^2 \), and equiconvergence with the trigonometric Fourier series of nonharmonic Fourier series with respect to system (1) in the interval \((-\pi, \pi)\). Uniform minimality of systems (1) in \( L^p \) and \( C \) is much less studied.

A sequence \( \{e_i\} \) of elements of a Banach space is said to be minimal if

\[ d_i \overset{df}{=} \text{dist}(e_i, \text{span}(e_k, \ k \neq i)) > 0, \]

for each \( i \); it is said to be uniformly minimal if \( d_i \geq \delta \|e_i\| \) for all \( i \), where \( \delta > 0 \) is independent of \( i \).

The latter property is of interest for several reasons. First, it is a necessary condition for the system \( \{(e_i)\} \) to be a basis. Next, Il'in [3] proved an important theorem, which can be stated as follows for systems (1) in the spaces \( L^p \). Let all \( \lambda_n \) lie in the horizontal strip \( |\text{Im } z| \leq H < \infty \), and let the number of these points in the rectangle \( t < \text{Re } z < t + 1, |\text{Im } z| \leq H \) be uniformly bounded with respect to \( t \in \mathbb{R} \). Next, let the system (1) be complete and minimal in \( L^p \), \( 1 \leq p < \infty \). Then the system (1) is uniformly minimal if and only if the nonharmonic Fourier series with respect to the system (1) of any \( f \in L^p \) is uniformly equiconvergent in \((-\pi, \pi)\) with the trigonometric Fourier series of \( f \).

We see that uniform minimality of exponential systems in \( L^p \) is a property of interest per se.

Suppose that all \( \lambda_n \) are real. Then \( \lambda_n - \lambda_m \to 0 \) obviously implies \( \|\exp(i\lambda_n t) - \exp(i\lambda_m t)\|_p \to 0 \). Hence the separability condition \( |\lambda_n - \lambda_m| \geq \delta > 0 \) is necessary for system (1) to be minimal in \( L^p \) and \( C \). We pose the following question: Do there exist complete minimal, but not uniformly minimal, systems of exponentials in \( L^p \) and \( C \) with real separable spectrum \( \Lambda = (\lambda_n) \)? In the present paper we show that the answer is "yes" by explicitly constructing such sequences.

Let us introduce some notation. Consider a real sequence

\[ M(\mu_n) \overset{\infty}{\leftarrow}, \quad \cdots < \mu_{-n} < \cdots < \mu_{-1} < \mu_0 \leq 0 < \mu_1 < \cdots < \mu_n < \cdots. \]

We write \( M_s = (\mu_n) \), \( -s \leq n \leq s \), \( s \in \mathbb{N} \). Thus, \( M_s \) is a finite symmetric collection of \( \mu_n \). By \( M_s + h \) we denote the translation of \( M_s \) by \( h \), that is, \( M_s + h = (\mu_n + h) \), \( -s \leq n \leq s \).

The main goal of this paper is to prove the following three theorems.
Theorem 1. Let $0 \leq \alpha_n \leq d$, $d < 1/4$, $n \in \mathbb{N}$, and
\[
\sum_{n=1}^{\infty} \frac{\alpha_n}{n} = +\infty.
\] (2)

Let $U$ be the sequence of all positive integers in the intervals $[2^s - s, 2^s + s]$, $s \geq 3$. Set
\[
\mu_n = n \text{ for } n \leq 0 \quad \text{and} \quad \mu_n = n - \alpha_n \text{ for } n > 0.
\] (3)

Then for
\[
\Lambda = \left( \mu_n : n \in \mathbb{Z} \setminus U \setminus \{0\} \right) \cup \left( \bigcup_{s=3}^{\infty} (M_s + 2^s) \right),
\] (4)
the system (1) is complete and minimal, but not uniformly minimal, in $L^1$.

Theorem 2. Let $\alpha_n$ and $U$ be the same as in Theorem 1 except that $d < 1/8$. Let
\[
\mu_0 = 0, \quad \mu_n = n + \frac{1}{2} \quad \text{for } n < 0, \quad \text{and} \quad \mu_n = n - \frac{1}{2} - \alpha_n \text{ for } n > 0.
\] (5)

Then for $\Lambda$ given by Eq. (4), the system (1) is complete and minimal, but not uniformly minimal, in $C$.

In the following $[z]$ stands for the greatest integer in $z$.

Theorem 3. Let $1 < p < \infty$, $1/p + 1/q = 1$. Let $V$ be the sequence of all integers in the intervals
\[
I_s = [2^s, 2^s + \lfloor \log s \rfloor], \quad s \geq 3.
\] (6)

Let
\[
\Lambda = (n : n < 0, n \in V) \cup (n - \frac{1}{q} : n \in \mathbb{N} \setminus V).
\]

Then the system (1) is complete and minimal, but not uniformly minimal, in $L^p$.

Note that the sequences $\Lambda$ occurring in Theorems 1–3 are real and separable.

§2. Lemmas. In the following statements, $L^\infty(-\pi, \pi)$ is understood to mean $C[-\pi, \pi]$.

Lemma 1. Suppose that the system
\[
\exp(i \mu_n t), \quad \mu_n \in M \subset \mathbb{R},
\] (7)
is not minimal in $L^p$, $1 \leq p \leq \infty$. If $\Lambda$ contains a subsequence of translations
\[
M_s + h_s, \quad s = s_j \to \infty, \quad h_s \in \mathbb{R},
\] then the system (1) is not uniformly minimal in $L^p$.

Proof. Since system (7) is not minimal in $L^p$, there exists a point $\mu = \mu_k \in M$ such that
\[
\text{dist}(\exp(i \mu t), \text{span}(\exp(i \mu_n t), \mu_n \in M, n \neq k)) = 0.
\]

It follows that the sequence
\[
\rho_s = \text{dist}(\exp(i \mu t), \text{span}(\exp(i \mu_n t), \mu_n \in M_s, n \neq k))
\]
satisfies
\[
\rho_s \to 0, \quad s \to \infty.
\] (8)

Set
\[
\rho_s(h) = \text{dist}(\exp(i(\mu + h)t), \text{span}(\exp(i \gamma_n t), \gamma_n \in M_s + h, \gamma_n \neq \mu + h)).
\]

Obviously,
\[
\rho_s(h) = \rho_s
\] (9)
for each $h \in \mathbb{R}$.

Let $s$ be so large that $\mu \in M_s$. Since $\Lambda$ contains the translation $M_s + h_s$, it follows that the distance from the function $\exp(i(\mu + h_s)t)$, which belongs to system (1), to the closure of the linear span of the other functions in this system does not exceed $\rho_s(h_s)$. However, by (9) and (8) we have $\rho_s(h_s) \to 0$ as $s = s_j \to \infty$. Thus, the cited distance tends to zero as $s = s_j \to \infty$. This means exactly that system (1) is not uniformly minimal in $L^p$. \qed

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