Left and Right Distributive Rings

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Abstract. By a distributive module we mean a module with a distributive lattice of submodules. Let A be a right distributive ring that is algebraic over its center and let B be the quotient ring of A by its prime radical H. Then B is a left distributive ring, and H coincides with the set of all nilpotent elements of A.

All rings are assumed to be associative and with a nonzero unit element and all modules are assumed to be unitary. By a distributive module we mean a module with distributive lattice of submodules. A module for which all its finitely generated submodules are cyclic is called a Bezout module. Expressions such as distributive ring mean that both the right and left conditions (here, distributivity) hold. In [1] it was proved that a right distributive domain algebraic over its center is left distributive. In [2] we gave an example of a right distributive domain that is not left distributive; this ring is a right Bezout ring, and all its maximal right ideals are ideals. The main results of the present paper are Theorems 1 and 2; moreover, we also prove other results concerning distributive rings and modules.

Theorem 1. Let A be a right distributive ring algebraic over its center. Then the quotient ring R of the ring A by its prime radical N is a left distributive ring and for each element x ∈ R there exists a positive integer n such that x^n R = Rx^n. Moreover, N coincides with the set of all nilpotent elements of the ring A.

Theorem 2. Let A be a right Bezout ring; assume that all maximal right ideals of A are ideals. Then the ring A is right distributive, and if either A is algebraic over its center and semiprime or A is a left Bezout ring, then A is a left distributive ring.

The proof of the theorems is decomposed into a series of lemmas. Let us introduce the necessary notation and definitions. An element of a ring is said to be regular if its right annihilator r(a) coincides with its left annihilator l(a) and is equal to zero. For a module M, denote by End(M), J(M), max(M), Lat(M), Sing(M), and sg(M) the endomorphism ring, the Jacobson radical, the set of all maximal submodules, the lattice of all submodules, the singular submodule, and the ideal of the ring End(M) formed by all endomorphisms with essential kernels, respectively. For a ring A, denote by C(A) and U(A), respectively, the center and the group of invertible elements of A. An element a of a ring A is said to be algebraic (integral) over C(A) if a is a root of a polynomial f(x) with coefficients from C(A), where the leading coefficient of the polynomial f(x) is regular (invertible) in A. The ring A is said to be algebraic (integral) over its center if all its elements are algebraic (integral) over C(A). A module M is said to be antisingular (singular) if Sing(M) = 0 (Sing(M) = M). A module is called a chain module if any two of its submodules are compatible with respect to inclusion. A module is said to be uniform if any two of its nonzero submodules have nonzero intersection. A module is said to be (Goldie) finite-dimensional if it contains no infinite direct sums of nonzero submodules. By a subquotient we mean a submodule of a quotient module. A module is said to be invariant (quasiinvariant) if all its submodules (all its maximal submodules) are its completely invariant submodules. A ring A is right quasiinvariant (right invariant) if all maximal right ideals (all right ideals) of the ring A are ideals if all cyclic right A-modules are quasiinvariant (invariant); A is right quasiinvariant if A/J(A) is right quasiinvariant. A prime ideal of a ring A not containing another prime ideals of the ring A is said to be a minimal prime ideal. A proper ideal is said to be completely prime if the quotient ring by this ideal is a domain, that is,
contains no zero divisors. A ring is said to be normal if all its idempotents are central. A ring without nonzero nilpotent elements is called a reduced ring. A subset \( T \) of a ring \( A \) is said to be:

(1) multiplicative if the subset \( T \) is closed under multiplication in \( A \), contains the unit element, and does not contain the zero of the ring \( A \);
(2) right commutative if for all elements \( a \in A \) and \( t \in T \) there are elements \( b \in A \) and \( u \in T \) such that \( au = tb \);
(3) right reversible if for all elements \( a \in A \) and \( t \in T \) such that \( ta = 0 \) there exists an element \( u \in T \) such that \( au = 0 \).

A subset \( T \) of a ring \( A \) is said to be a set of right denominators if it satisfies the following two equivalent conditions:

(1) \( T \) is a right commutative right reversible multiplicative subset;
(2) there exists a ring \( A_T \) and a ring homomorphism \( f_T : A \to A_T \) such that \( f(T) \subseteq U(A_T) \),

\[ A_T = \{ f(a)f(t)^{-1} \mid a \in A, t \in T \}, \]

and

\[ \text{Ker}(f) = \{ a \in A \mid at = 0 \text{ for some } t \in T \}. \]

Under these assumptions, the ring \( A_T \) is called a right ring of quotients of the ring \( A \) with respect to \( T \) and \( f_T \) is called the canonical homomorphism; for each element \( a \in A \) we write \( a_T \) instead of \( f_T(a) \). For each right ideal \( B \) of the ring \( A \) we denote by \( B_T \) the right ideal of the ring \( A_T \) generated by the set \( f_T(B) \). If \( T = A \setminus M \), where \( M \) is a right ideal of the ring \( A \), then \( f_M, A_M, a_M, \) and \( B_M \) stand for \( f_T, A_T, a_T, \) and \( B_T \). A ring \( A \) is said to be right localizable, if for each its maximal right ideal \( M \) there exists a right ring of quotients \( A_M \). A left ring of quotients \( \rho_A \) with respect to a set \( T \) of left denominators and a canonical homomorphism \( \rho_f : A \to \rho_A \) are defined similarly. If \( T = A \setminus M \), where \( M \) is a left ideal of a ring \( A \), then we also write the subscript \( M \) instead of \( T \). Left localizable rings are defined similarly to right localizable rings. An endomorphism \( g \) of a module \( M \) is said to be locally nilpotent if for each element \( m \in M \) there exists a positive integer \( n = n(m) \) such that \( g^n(m) = 0 \). A submodule \( N \) of a module \( M \) is said to be closed if each submodule \( N_1 \subseteq \text{Lat}(M) \) that is an essential extension of the module \( N \) coincides with \( N \). Submodules \( N \) and \( T \) of a module \( M \) are said to be comaximal if \( N + T = M \). A submodule \( N \) of a module \( M \) is said to be small if the relation \( N + T = M \), where \( T \subseteq \text{Lat}(M) \), implies \( T = M \). A module is called a semi-Noetherian (semi-Artinian) if each its nonzero subquotient possesses a simple quotient module (submodule).

**Lemma 1.** Let \( T \) be a right commutative multiplicative subset of a ring \( A \). Then we have the following.

(1) If \( T \) is weakly right reversible, then \( T \) is a set of right denominators.
(2) In the ring \( A \) each element with zero square commutes with each element of the set \( T \), then \( T \) is a set of right denominators.
(3) If for each element \( a \in A \) there exists a positive integer \( n = n(a) \) such that \( r(t^n) = r(t^{n+1}) \), then \( T \) is a set of right denominators.
(4) If \( A \) is a ring with the ascending chain condition for right annihilators, then \( T \) is a set of right denominators.

**Proof.** (1) Let \( a \in A \) and \( t \in T \); assume that \( b \equiv at \), where \( ta = 0 \). Then \( b^2 = 0 \) and \( tb = 0 \) and, by assumption, \( bx = 0 \) for some \( x \in T \). Assume that \( u \equiv tx \in T \). Then \( au = 0 \) and \( T \) is right reversible.

(2) The assertion follows from item (1).

(3) Let \( a \in A \) and \( t \in T \); assume that \( ta = 0 \). By assumption, there exists a positive integer \( n \) such that \( r(t^n) = r(t^{n+1}) \). Since the set \( T \) is right commutative and multiplicative, there exist elements \( b \in A \) and \( u \in T \) such that \( t^nb = au \). Since \( t^{n+1}b = ta = 0 \), we have \( b \in r(t^{n+1}) = r(t^n) \). Therefore, \( au = t^nb = 0 \) and the set \( T \) is right reversible. Then \( T \) is a set of right denominators.

(4) Let \( a \in A \) and \( t \in T \); assume that \( ta = 0 \). Since \( A \) is a ring with ascending chain condition for right annihilators and since \( r(t^i) \subseteq r(t^{i+1}) \) for each positive integer \( i \), there exists a positive integer \( n \) such that \( r(t^n) = r(t^{n+1}) \). By item (3), \( T \) is a set of right denominators. \( \square \)