Properties of Morse Forms that Determine Compact Foliations on $M^2_g$

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In [1, 2] P. Arnoux and G. Levitt showed that the topology of the foliation of a Morse form $\omega$ on a compact manifold is closely related to the structure of the integration mapping $[\omega]: H_1(M) \to \mathbb{R}$. In this paper we consider the foliation of a Morse form on a two-dimensional manifold $M^2_g$. We study the relationship of the subgroup $\text{Ker}[\omega] \subset H_1(M^2_g)$ with the topology of the foliation. We consider the structure of the subgroup $\text{Ker}[\omega]$ for a compact foliation and prove a criterion for the compactness of a foliation.

§1. Preliminary definitions

Consider a closed form $\omega$ with Morse singularities on $M^2_g$. This form determines a foliation $\mathcal{F}$ on $M^2_g \setminus \text{Sing } \omega$.

Let us define a foliation with singularities $\mathcal{F}_\omega$ on $M^2_g$ as follows.

Suppose that the foliation $\mathcal{F}$ is locally (in a sufficiently small neighborhood of a singular point $p \in \text{Sing } \omega$) determined by the levels of a function $f_p$ such that $f_p(p) = 0$.

Definition 1. A nonsingular leaf of a foliation $\mathcal{F}_\omega$ is a leaf $\gamma \in \mathcal{F}$ such that $\gamma \cap f_p^{-1}(0) = \emptyset$ for all $p \in \text{Sing } \omega$.

Put $F_p = p \cup \{\gamma \in \mathcal{F} | \gamma \cap f_p^{-1}(0) \neq \emptyset\}$. Also put $F = \bigcup_{p \in \text{Sing } \omega} F_p$.

Definition 2. A singular leaf of a foliation $\mathcal{F}_\omega$ is a connected component of $F$.

There is only a finite number of singular leaves (because the form is Morse).

A foliation $\mathcal{F}_\omega$ is called compact if all its leaves are compact.

A closed form $\omega$ determines the mapping $[\omega]: H_1(M^2_g) \to \mathbb{R}$ (integration over cycles). The image of this mapping $\text{Im}[\omega]$ represents the period group of the form $\omega$. Note that $\text{rk } \text{Im}[\omega] = \text{dirr } \omega + 1$, where $\text{dirr } \omega$ is the degree of irrationality of the form $\omega$.

If $\text{dirr } \omega \leq 0$, then the foliation $\mathcal{F}_\omega$ is compact [3]. If $\text{dirr } \omega \geq g$, then the foliation $\mathcal{F}_\omega$ has a noncompact leaf [4]. If $0 < \text{dirr } \omega < g$, then the foliation can be compact as well as noncompact. The study of the subgroup $\text{Ker}[\omega]$ yields a condition for the compactness of a foliation in the latter case also.

Consider the intersection operation of 1-cycles

$$\varphi: H_1(M^2_g) \times H_1(M^2_g) \to \mathbb{Z}.$$  

This operation is a nondegenerate skew-symmetric bilinear mapping.

By $\varphi_\omega$ denote the restriction of the mapping $\varphi$ to the subgroup $\text{Ker}[\omega] \subset H_1(M^2_g)$:

$$\varphi_\omega: \text{Ker}[\omega] \times \text{Ker}[\omega] \to \mathbb{Z}.$$  

Obviously, $\text{rk } \text{Ker } \varphi_\omega \leq \text{rk } \text{Ker}[\omega] = 2g - (\text{dirr } \omega + 1)$. For small values of $\text{dirr } \omega$ a sharper estimate exists.

Proposition 1. $\text{rk } \text{Ker } \varphi_\omega \leq \text{dirr } \omega + 1$.  

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Proof. Suppose that \( \text{Ker} \varphi_\omega = (z_1, \ldots, z_k) \). By \( Dz_i \) denote the cycles dual to \( z_i \); then \( Dz_i \circ z_j = \delta_{ij} \). Assume that \( \sum n_i \int_{Dz_i} \omega = 0 \) and put \( z = \sum n_i Dz_i \). We see that \( z \in \text{Ker}[\omega] \) and \( z \circ z_j = n_j \). All the \( n_j \) equal zero because \( z_j \in \text{Ker} \varphi_\omega \). Thus, the integrals \( \int_{Dz_i} \omega \) are linearly independent over \( \mathbb{Q} \) and \( \text{dirr} \omega \geq k - 1 \). The proposition is proved. \( \square \)

A subgroup \( H \subset H_1(M^2_\omega) \) is called isotropic with respect to cycle intersection if \( z \circ y = 0 \) for all \( z, y \in H \). An isotropic subgroup \( H \) is maximal if for all \( z \notin H \) there exists a \( y \in H \) such that \( z \circ y \neq 0 \).

Note that the subgroup \( \text{Ker} \varphi_\omega \) is isotropic.

**Proposition 2.** Let \( H_0 \) be a maximal isotropic subgroup in the group \( \text{Ker}[\omega] \). Then

\[
\text{rk} H_0 = \frac{1}{2}(\text{rk Ker}[\omega] + \text{rk Ker} \varphi_\omega).
\]

**Proof.** Since the mapping \( \varphi_\omega \) is a symplectic form on \( \text{Ker}[\omega] \otimes \mathbb{R} \), the statement follows by splitting the symplectic space \( \text{Ker}[\omega] \otimes \mathbb{R} \) into an orthogonal direct sum of two-dimensional nondegenerate and one-dimensional degenerate subspaces. \( \square \)

To each nonsingular leaf \( \gamma \in F_\omega \) assign its homology class \( [\gamma] \). The image of all compact nonsingular leaves under this correspondence generates a subgroup in \( H_1(M^2_\omega) \); denote it by \( H_\omega \). Note that the subgroup \( H_\omega \) is isotropic and \( H_\omega \subset \text{Ker}[\omega] \).

\section{Properties of Morse forms that determine compact foliations}

**Theorem 1.** Suppose that a foliation on \( M^2_\omega \) is determined by a Morse form \( \omega \). If the foliation \( F_\omega \) is compact, then

\[
\text{rk Ker} \varphi_\omega = \text{dirr} \omega + 1.
\]

**Proof.** Nonsingular compact leaves of \( F_\omega \) generate a subgroup \( H_\omega \subset H_1(M^2_\omega) \). There is also a finite number of singular leaves, say, \( \gamma_1^0, \ldots, \gamma_k^0 \). Consider the embeddings \( j_s: \gamma_s^0 \to M^2_\omega \), \( s = 1, \ldots, k \), and the induced homology mappings \( j_s*: H_1(\gamma_s^0) \to H_1(M^2_\omega) \). For each group \( H_1(\gamma_s^0) \), choose its maximal subgroup \( Z_s \) so that the image \( j_s*Z_s \subset H_1(M^2_\omega) \) is an isotropic subgroup.

Consider the subgroup \( H_0 = (H_\omega, j_s*Z_s, s = 1, \ldots, k) \), which is obviously isotropic. Moreover, we have \( H_0 \subset \text{Ker}[\omega] \).

Consider a cycle \( z \in H_1(M^2_\omega) \) such that \( z \circ H_0 = 0 \); then \( z \circ H_\omega = 0 \). In the paper [5] it was shown (see the proof of Theorem 1.2) that the cycle \( z \) can be realized by a curve \( \alpha \subset \bigcup \gamma_s^0 \), provided that the foliation \( F_\omega \) is compact and \( z \circ H_\omega = 0 \). Let us assume that \( \alpha = \sum \alpha_s \), where \( \alpha_s \subset \gamma_s^0 \). We have \( z \circ j_s*Z_s = 0 \) for all \( s \). Hence, \( [\alpha_s] \circ j_s*Z_s = 0 \) because \( \alpha_p \cap \gamma_s^0 = \emptyset \) whenever \( p \neq s \). Since \( \alpha_s \subset \gamma_s^0 \), we get \( [\alpha_s] \in \text{Im} j_s* \). Thus, \( [\alpha_s] \in j_s*Z_s \) and \( z \in H_0 \) (by the construction of the group \( Z_s \)).

So \( H_0 \) is a maximal subgroup of \( H_1(M^2_\omega) \). Thus, \( \text{rk} H_0 = g \) (this is shown in [5]). On the other hand, \( H_0 \) is a maximal subgroup of \( \text{Ker}[\omega] \), and Proposition 2 implies that

\[
\text{rk} H_0 = \frac{1}{2}(\text{rk Ker}[\omega] + \text{rk Ker} \varphi_\omega).
\]

Since \( \text{dirr} \omega = 2g - 1 - \text{rk Ker}[\omega] \), the theorem is proved. \( \square \)

**Remark.** The converse statement is not true, i.e., there exist noncompact foliations such that the previous relation holds.

\section{A criterion for the compactness of a foliation}

In [4] it is shown that if there are \( g \) compact leaves that are homologically independent, then all the leaves are compact. Taking into account the structure of the subgroup \( \text{Ker}[\omega] \), let us strengthen this criterion of compactness. Let us consider the intersection \( H_\omega \cap \text{Ker} \varphi_\omega \).