Obstructions to Splitting Manifolds with Infinite Fundamental Group

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ABSTRACT. In this paper we calculate the obstruction groups to splitting along one-sided submanifolds when the fundamental group of the submanifold is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. We also consider the case where the obstruction group is not a Browder-Livesey group. We construct a new Levine braid that connects the Wall groups to the obstruction group for splitting. We solve the problem of the mutual disposition of images of several natural maps in Wall groups for finite 2-groups with exceptional orientation character.

Key words: splitting of manifolds, obstruction group, Wall group, Browder-Livesey group, Levine braid, universal repelling square.

§1. Introduction

Let $X \subset Y$ be a pair of smooth or piecewise smooth manifolds, let $U$ be a tubular neighborhood of $X$ in $Y$, $n = \dim X \geq 5$,

$$
\Phi = \begin{pmatrix}
\pi_1(\partial U) & \pi_1(Y \setminus X) \\
\pi_1(X) & \pi_1(Y)
\end{pmatrix}
$$

be the universal repelling square of fundamental groups. Then the groups $LS_n(\Phi)$, which functorially depend on the square $\Phi$ and $n \mod 4$, define an obstruction group for splitting of simple homotopic equivalences $f: M \to Y$ along the submanifold $X$. In the general case, the groups $LS(\Phi)$ as cobordism groups were defined by Walter in [1] (see also [2]).

For the case in which horizontal maps in the square $\Phi$ are epimorphisms, the algebraic definition for groups $LS(\Phi)$ of simply-connected submanifold is presented in [3]. In this case the universal repelling square of the fundamental groups $\Phi$ is said to be its geometric diagram.

In this paper we consider the case where $X$ is a one-sided submanifold of codimension 1 in $Y$. If the embedding $X \subset Y$ induces an isomorphism of fundamental groups, then the pair $(Y, X)$ is called a Browder-Livesey pair. In this case the group $LS_n(\Phi)$ coincides with the group $L_{Nn}(\pi \to G)$, where $\pi = \pi_1(Y \setminus U)$, $G = \pi_1(Y)$, $\pi \to G$ is an embedding of index 2, and the group $L_{Nn}(\pi \to G)$ in the algebraic sense is a Wall group for a ring with antistructure (see [1, 4]). We shall estimate several obstruction groups to splitting for geometric squares in which the fundamental group $\pi_1(X)$ for the submanifold $X$ of codimension 1 is the infinite Abelian group $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. Let us remark that in the case of infinite groups we know very little about obstruction groups to splitting at present. The following facts explain this situation. First, even for the simplest infinite groups $\pi$, the Wall groups $L(\pi)$ are infinitely generated. Secondly, the algebraic definition of the groups $LS$ in the general case is still missing (see [1-5]).

In this paper, as an immediate consequence of our calculations, we describe several maps of Wall groups important for geometric applications: the transfer map (corresponding to passing to the double covering) and the induced map for embeddings of infinite index 2 groups that appear in the square $\Phi$. Also we consider examples where geometric diagrams appear.

In [6], in order to solve the problem of realizability of Wall group elements by normal maps of closed manifolds, the notion of type of element was introduced and the classification by type of elementary 2-groups was performed. The determination of the type of element is based on the two-string diagram.
connecting the groups \( L(\pi), L(G) \) and \( LN(\pi \to G) \) for a subgroup \( \pi \) of index 2 in the group \( G \) (see [7, 8]). Even when all the maps in the two-string diagram are known, difficulties in the classification still remain, because the knowledge of the mutual disposition of the images

\[ i_*: L_n(\pi) \to L_n(G), \quad ti_*: LN_n(\pi \to G) \to L_n(G). \]

in the group \( L_n(G) \) is required. Difficulties of this sort did not allow to finish the classification by type in the case of finite Abelian 2-groups in [9]. Note that this problem is closely connected with the problem of identifying Wall group elements and with the calculation of for all differentials in the exact surgery sequence (see [10]), which was introduced in [11].

In this paper, we apply results about the groups \( LS_n(\Phi) \) to determine the mutual disposition of the images of \( i_* \) and \( ti_* \) and to classify the groups \( L_n(G) \) for all \( n = 0, 1, 2, 3 \mod 4 \), by element type for an arbitrary finite Abelian 2-group \( G \) with exceptional orientation character (i.e., the group \( G \) contains the direct summand \( \mathbb{Z}/2 \) on which the orientation character is nontrivial). This result is due to Yu. V. Muranov. To this end, we construct a diagram which connects Wall groups and obstruction groups for splitting. This diagram gives another algebraic calculation method and it can be applied in the more general case than the one considered in this paper. This diagram is similar to the diagrams introduced in [1, 7, 8]. The application of this diagrams has led to some progress in the calculation of \( L \)-groups and natural maps between them (see [9, 10, 12]).

All the groups \( L, LN, LS \) under consideration are assumed to be endowed with a decoration \( S \); in geometry this corresponds to surgeries up to simple homotopy equivalence (see [1]).

\section*{§2. Diagrams and exact sequences for the groups \( LS_n(\Phi) \)}

Denote the fundamental groups and the maps from the geometric diagram (1) as follows:

\[
\Phi = \begin{pmatrix}
A^+ & \longrightarrow & C^+ \\
\downarrow r & & \downarrow i \\
B^- & \longrightarrow & D^+
\end{pmatrix}
\]

All the groups from this square are endowed with homomorphisms into the group \( \mathbb{Z}/2 \) (the orientation character). All the homomorphisms expect \( j \) commute with the orientation homomorphism, the homomorphism \( j \) commutes with it on the image of the map \( r \) and does not commute outside this image. We denote this by using the upper index "±" for the corresponding groups. In a geometric diagram, the maps \( r \) and \( l \) are embeddings onto subgroups of index 2, the maps \( s, j \) are epimorphisms. If \( t \in B \setminus r(A) \), then \( j(t) \in D \setminus l(C) \). This is why we can denote by "−" the orientation of \( D \) such that the value of the orientation homomorphism is compatible with the orientation of \( B \) induced by the homomorphism \( j \).

It is important to note that in this case the homomorphism \( l \) remains compatible with orientations. We shall omit the index "+" when indicating the orientation for the corresponding groups.

There is a close relationship between the groups \( LS_n(\Phi) \) and the \( L \)-groups of all groups from the square \( \Phi \). We have the following commutative diagram (Levine braid) from [1]:

\[
\begin{array}{cccc}
& L_n(B^-) & \longrightarrow & L_{n+1}(C \to D) \\
LS_n & & \downarrow & \\
& L_{n+2}(\Phi) & \longrightarrow & LN_{n-1}(A \to B) \\
& & \downarrow & \\
& LN_{n-1}(A \to B) & \longrightarrow & LN_n(A \to B)
\end{array}
\]

(2)

In [7, 8], the following two-row diagram was introduced for the case of Browder–Livesey groups, i.e., when the maps \( s, j \) in the square \( \Phi \) are isomorphisms:

\[
\begin{array}{cccc}
L_n(A) & \longrightarrow & L_n(B) & \longrightarrow & LN_{n-2}(A \to B) & \longrightarrow & L_{n-2}(B^-) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
LN_{n-1}(A \to B) & \longrightarrow & LN_{n-1}(B^-) & \longrightarrow & L_{n-1}(A) & \longrightarrow & LN_{n-2}(A \to B)
\end{array}
\]

(3)

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