The Spectrum of the Coriolis Operator in Axisymmetric Domains with Edges

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Let $G$ be a bounded domain in $\mathbb{R}^3$, and let $H$ be the Hilbert space of vector functions $\vec{U} = (u_1, u_2, u_3)$, $u_i \in L^2(G)$, $i = 1, 2, 3$, with the inner product defined by

$$(\vec{U}, \vec{V}) = \int_G (u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3) \, dxdydz.$$ 

Furthermore, let $S$ be the closure in $H$ of the linear manifold

$$\{ \vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \mid \varphi_i \in C^\infty(G), \ i = 1, 2, 3, \ \nabla \cdot \vec{\varphi} = 0 \}$$

of solenoidal vectors, and let $P$ be the operator of orthogonal projection on the subspace $S$. We define a linear operator $B: S \to S$ by setting $B\vec{U} = P(\vec{U} \times \vec{k})$, where $\vec{k} = (0, 0, 1)$. Obviously, $B$ is bounded and skew-Hermitian. In the literature, $B$ is often called the gyroscopic or Coriolis operator, since the equation

$$\frac{\partial \vec{U}}{\partial t} = B\vec{U}, \quad \vec{U}(0) = \vec{U}_0 \in S \quad (1)$$

describes small oscillations of ideal incompressible fluid that completely fills a container $G$ and rotates about the vertical axis $Oz$ at the constant angular velocity $\omega = 1/2$. This problem was studied by Sobolev in the famous paper [1]. Ralston [2] proved a general theorem on the position of the spectrum of the Coriolis operator stating that $\sigma(iB) = [-1, 1]$ for any domain $G$. It should be pointed out that the qualitative structure of the spectrum of $B$ depends on the configuration of $G$. On the other hand, the properties of the solutions to the Cauchy problem (1) (such as almost-periodicity in the time variable $t$) are closely related to the structure of the spectrum. Almost-periodicity of the model problem corresponding to (1) for the case of two space variables, was studied by numerous authors (see the reviews [3, 4]). The three-dimensional problem is much less studied. The structure of the spectrum of the Coriolis operator...
is completely studied only for two container shapes, namely, a right circular cylinder and an ellipsoid of revolution [5]. In both cases, the operator $iB$ has a complete system of eigenfunctions, and the set of its eigenvalues is everywhere dense in the interval $[-1, 1]$; it follows that all solutions of the corresponding initial-boundary value problem are almost periodic in time. In [6] it was proved that there exists a toroidal domain in which the solutions of (1) are not almost periodic.

In what follows, we consider domains axisymmetric with respect to the rotation axis. It is readily shown that in this case the space $S$ can be represented as an infinite sum of orthogonal subspaces $S_k$ invariant with respect to $B$. We prove that the spectrum of each of the operators $iB_k := iB|_{S_k}$, $k = 0, \pm 1, \pm 2, \ldots$, which are the restrictions of $iB$ to the subspaces $S_k$, also coincides with the entire interval $[-1, 1]$. Using this fact and the method of [7] (see also [8, 9]) we show that for some axisymmetric domains with edges (e.g., for a cone) the spectrum of $iB_0$ is continuous in some subintervals of the interval $[-1, 1]$. This means that the eigenfunctions of the original operator $B$ (if they exist) do not form a basis in $S$, and in such domains problem (1) necessarily has solutions that are not almost periodic in time.

In what follows, we assume that $G$ is a domain axisymmetric with respect to the axis $Oz$, bounded by a piecewise-smooth surface $\partial G$ and satisfying the cone condition.

Let $\tilde{W}_2^1 := W_2^1(G) \ominus \{1\}$ be the subspace of the Sobolev space $W_2^1(G)$ orthogonal to constants. Let $(r, \varphi, t)$ be the cylindrical coordinate system related to $(x, y, z)$ by $x = r \cos \varphi$, $y = r \sin \varphi$, $z = t$, and let $D$ be the domain in the plane $\varphi = 0$ that sweeps out $G$ in revolving around the axis $Ot$. The cited change of coordinates transforms $H$ and $\tilde{W}_2^1$ into spaces that we denote by $H_{(r)}(D)$ and $\tilde{W}_{(r),1}^1(D)$, respectively. These are Hilbert spaces of classes of measurable functions on the cylinder $P = D \times (0, 2\pi)$, with the inner products

$$
\langle \bar{U}(r, \varphi, t), \bar{V}(r, \varphi, t) \rangle_{H_{(r)}(D)} := \iint_P \sum_{i=1}^3 u_i(r, \varphi, t) \bar{u}_i(r, \varphi, t) r \, dr \, d\varphi \, dt,
$$

and

$$
\langle \bar{h}(r, \varphi, t), \bar{g}(r, \varphi, t) \rangle_{\tilde{W}_{(r),1}^1(D)} = \iint_P \left\{ \bar{h} \bar{g} + \frac{\partial h}{\partial r} \frac{\partial \bar{g}}{\partial r} + \frac{\partial h}{\partial \varphi} \frac{\partial \bar{g}}{\partial \varphi} + \frac{1}{r^2} \frac{\partial h}{\partial \varphi} \frac{\partial \bar{g}}{\partial \varphi} \right\} r \, dr \, d\varphi \, dt.
$$

Let $L_{(r)}(D)$ and $W_{(r),1}^1(D)$, $k = 0, \pm 1, \pm 2, \ldots$, denote the Hilbert spaces of classes of measurable functions on $D$ equipped with the norms

$$
\|f(r, t)\|_{L_{(r)}(D)}^2 = \int_D |f(r, t)|^2 r \, dr \, dt,
$$

and

$$
\|g(r, t)\|_{W_{(r),1}^1(D)}^2 = \int_D \left\{ |g(r, t)|^2 + \left| \frac{\partial g}{\partial r} \right|^2 + \left| \frac{\partial g}{\partial \varphi} \right|^2 + \frac{k^2}{r^2} |g(r, t)|^2 \right\} r \, dr \, dt.
$$

Theorem 1. The following representations hold:

$$
\tilde{W}_{(r),1}^1(P) = \bigoplus_{k=0, \pm 1, \pm 2, \ldots} \bigoplus_{k=0, \pm 1, \pm 2, \ldots} W_{(r),1}^1(D) e^{ik\varphi} \bigoplus \{ W_{(r),0}^1(D) \ominus \{1\} \}, \quad H_{(r)}(D) = \bigoplus_{k=-\infty}^{+\infty} H_{(k, r)}.
$$

Here the elements of the spaces $H_{(k, r)}$ have the form

$$
H_{(k, r)} := \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1(r, t) \\ f_2(r, t) \\ f_3(r, t) \end{pmatrix} e^{ik\varphi} : f_i(r, t) \in L_{(r)}(D), i = 1, 2, 3 \right\},
$$

and the sums are orthogonal with respect to the inner products (3) and (2), respectively.

Let the cylindrical change of coordinates transform $S$ to the space $S_{(r)}$, let $P_k$ be the operators of orthogonal projection of $H_{(r)}$ onto the subspaces $H_{(k, r)}$ (in the sense of (2)), and let the orthogonal matrix in the previous expression be denoted by $T_\varphi$. 227