Coedge Regular Graphs without 3-Stars

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ABSTRACT. We describe coedge regular graphs such that antineighborhoods of their vertices are coedge regular graphs with the same value of the parameter $\mu$. As a consequence of the main theorem, we obtain a classification of coedge regular graphs without 3-stars.

KEY WORDS: coedge regular graph, extensions by complete graphs, antineighborhood, graphs without 3-stars.

We consider nonoriented graphs without loops and multiple edges. For a vertex $a$ of a graph $\Gamma$, by $[a]$, respectively $[a]^\prime$, we denote the subgraph induced by $\Gamma$ on the set of all vertices adjacent to $a$, respectively of all vertices that are not adjacent to $a$ (and that differ from $a$). This subgraph is called the neighborhood, respectively the antineighborhood, of the vertex $a$.

A graph $\Gamma$ is said to be a regular graph of valency $k$ if for each vertex $a$ of $\Gamma$, the neighborhood $[a]$ consists of exactly $k$ vertices. Further, $\Gamma$ is called a coedge regular graph with parameters $(v, k, \mu)$ if $\Gamma$ is a regular $k$-valent graph with $v$ vertices and for each couple $a, b$ of nonadjacent vertices there are exactly $\mu$ vertices adjacent both to $a$ and $b$, i.e., $|[a] \cap [b]| = \mu$.

In this paper we classify coedge regular graphs such that antineighborhoods of their vertices are coedge regular graphs with the same value of the parameter $\mu$.

By $(m, n)$ denote the complete bichromatic graph with monochromatic parts of order $m$ and $n$. The graph $(m, 1)$ is called an $m$-star whenever $m \geq 2$. We prove below (see Lemma 1) that the class of graphs that we consider is contained in the extensive class of graphs without 3-stars. Further, the main theorem of the paper leads to a classification of coedge regular graphs without 3-stars.

A graph on $X \times Y$ is called an $m \times n$-graph if $|X| = m$, $|Y| = n$, and the vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if either $x_1 = x_2$ or $y_1 = y_2$. An $m \times n$ graph is also called a rectangular lattice. Such a graph with $m = n$ is called a lattice graph and is denoted by $L_2(m)$. Further, a triangle graph $T(m)$ is a graph whose vertices are unordered couples of elements of $X$, $|X| = m$, and the couples $\{a, b\}$ and $\{c, d\}$ are connected by an edge if and only if they have a common element. A coedge regular graph with parameters $(v, k, \mu)$ is said to be a strictly regular graph with parameters $(v, k, \lambda, \mu)$ if for each couple of adjacent vertices $a, b$ there are exactly $\lambda$ vertices adjacent both to $a$ and $b$, i.e., $|[a] \cap b]\| = \lambda$. Let $A$, $B$, and $C$ be three sets. Let us say that $B$ and $C$ split $A$ if $A \subseteq B \cup C$ and $B \cap C$ contains no elements of $A$. In what follows, a subgraph of $\Gamma$ means an induced subgraph. Finally, the $\alpha$-extension of $\Gamma$ is the graph obtained by replacing each vertex $a \in \Gamma$ by an $\alpha$-clique $(a)$ (an $\alpha$-clique is the complete graph on $\alpha$ vertices); vertices of $(a)$ and $(b)$ are adjacent if and only if $a$ and $b$ are adjacent in $\Gamma$.

Theorem. Let $\Gamma$ be a coedge regular graph such that the antineighborhoods of all its vertices are coedge regular graphs with the same value of the parameter $\mu$. Then $\Gamma$ is an $\alpha$-extension ($\alpha \geq 1$) of one of the following graphs:

1. a completely disjoint graph with $m$ vertices ($m \neq 2$);
2. an $m \times n$ rectangular lattice ($m \geq 3$, $n \geq 3$) or a triangle graph $T(m)$ ($m \geq 6$);
3. the Schlafli graph (the strictly regular graph with parameters $(27, 16, 10, 8)$ that is the complementary graph of the point graph of the generalized quadrilateral $GQ(2, 4)$).

Corollary. A coedge regular graph without 3-stars is either the complementary graph of a regular graph without triangles or one of the graphs described in the theorem.

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Proof. Let $\Gamma$ be a coedge regular graph that is not a complementary graph to a regular graph without triangles. Then for each vertex $a \in \Gamma$ there are nonadjacent vertices in $[a]'$ and for each couple $b, c$ of nonadjacent vertices in $[a]'$ the inclusion $[b] \cap [c] \subset [a]$ holds. In fact, otherwise there would be a vertex $d$ in $[a] \cap [b] \cap [c]$ and a subgraph on the set $\{a, b, c, d\}$ that is a 3-star in $\Gamma$. This contradicts the assumption. 

Proof of the theorem. Let $\Gamma$ be a coedge regular graph that is a counterexample for the theorem and has the smallest possible number of vertices. If $\Gamma$ does not contain nonadjacent vertices, then $\Gamma$ is a complete graph and the value of $\mu$ is not defined, but in this case the antineighborhood of each vertex is the empty graph. If $\Gamma$ contains nonadjacent vertices and $\mu = 0$, then $\Gamma$ is a union of $m$ complete subgraphs, $m \geq 2$. In this case the antineighborhood of a vertex is a union of $m - 1$ complete subgraphs and $m - 1 \geq 2$. Thus, $\Gamma$ is a coedge regular graph with parameters $(v, k, \mu)$ and $\mu > 0$. If $\mu = k$, then the antineighborhood of a vertex is a coclique. So $\mu = 0$ and $\Gamma$ is a completely disjoint graph. Consequently, $0 < \mu < k$.

Lemma 1. If $a, b$ are adjacent vertices of $\Gamma$, then $[a] \cap [b]'$ and $[a]' \cap [b]$ are cliques. Moreover, $\Gamma$ contains no 3-stars.

Proof. Let $c$ and $d$ be nonadjacent vertices of $[a] \cap [b]'$. By assumption, $[c] \cap [d] \subset [a]'$. However, this contradicts the fact that $[c] \cap [d]$ contains the vertex $a \in [b]$.

Let $\Delta = \{a, b, c, d\}$ be a 3-star in $\Gamma$ and $a$ be a three-valent vertex of $\Delta$. Then $[a] \cap [b]'$ contains nonadjacent vertices $c$ and $d$. This contradicts the previous statement. 

Lemma 2. Let $a$ and $b$ be nonadjacent vertices of $\Gamma$.

1. If $c \in [a]' \cap [b]'$, then $[c]$ does not intersect $[a] \cap [b]$.
2. If $w \in [a] \cap [b]' \cap [w]'$ and $[a] \cap [b]' \cap [w]'$ are cliques.

Proof. By assumption, $[a] \cap [b] \subset [c]'$. This proves the first statement. By Lemma 1, $[w] \cap [b]'$ and $[a] \cap [w]'$ are cliques. So we have the second statement.

Lemma 3. If $\Gamma$ is a strictly regular graph, then the smallest eigenvalue of $\Gamma$ is not equal to $-2$.

Proof. Suppose that the smallest eigenvalue of $\Gamma$ equals $-2$. By the Seidel Theorem [1], $\Gamma$ is one of the following graphs: a triangle graph $T(m)$, a lattice graph $L_2(m)$, a Shrikhande graph with the same parameters as the graph $T(8)$, the Petersen graph, a Clebsch graph, and a Schl"afli graph. However, Shrikhande and Petersen graphs contain 3-stars. In a Clebsch graph, the antineighborhood of a vertex is the 5-clique. In Chang graphs, the antineighborhoods of vertices are not coedge regular. The other graphs do not contradict the theorem. The lemma is proved.

Consider nonadjacent vertices $a$ and $b$. By $\Lambda_{a,b}$ denote the subgraph $[a]' \cap [b]'$ of the graph $\Gamma$. Put $\beta = |\Lambda_{a,b}|$. Then $\beta$ does not depend on the choice of nonadjacent vertices $a, b$, and $\beta = v - 2k + \mu - 2$.

Lemma 4. Suppose that $\Lambda_{a,b}$ is a clique. Then $\beta = k - 2\mu + 1$ and $\Lambda_{x,y}$ is a clique for any two nonadjacent vertices $x, y$ of the graph $\Gamma$.

Proof. If $e \in \Lambda_{a,b}$, then, by Lemma 2, $[e]$ contains $\mu$ vertices of $[a] \cap [b]'$ and $\mu$ vertices of $[a]' \cap [b]$. So $k = 2\mu + (\beta - 1)$, i.e., $\beta = k - 2\mu + 1$.

Let $x, y$ be nonadjacent vertices and $z$ belong to $\Lambda_{x,y}$. Then $[z]$ contains $\mu$ vertices of $[z] \cap [y]'$ and $\mu$ vertices of $[z]' \cap [y]$; the other $k - 2\mu$ vertices of $[z]$ belong to $\Lambda_{x,y}$. Hence, $\Lambda_{x,y}$ is a clique. The lemma is proved.

In Lemmas 5–14 we suppose that $\Lambda = \Lambda_{a,b}$ is a clique with $\beta$ vertices, $\beta = k - \mu + 1$. We say that a vertex $c \in ([a] \cap [b]) \cup ([a]' \cap [b])$ is an $i$-point for $\Lambda$ if $[c] \cap \Lambda = i$. Note that if an $i$-point $c$ for $\Lambda$ lies in $[a] \cap [b]'$, then $c'$ does not contain $i$ points of $[a] \cap [b]'$. In fact, $\Lambda_{b,c}$ contains $\beta - i$ points of $\Lambda$ and $i$ points of $[a] \cap [b]'$. 

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