Behavior of Solutions of Quasilinear Elliptic Inequalities in an Unbounded Domain

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ABSTRACT. We consider the solutions of the inequality \( Lu \leq \varphi(|\nabla u|) \), where \( L \) is a uniformly elliptic homogeneous operator and \( \varphi \) is a function increasing faster than any linear function but not faster than \( \xi \ln \xi \), in the unbounded domain

\[
\left\{ x \in \mathbb{R}^n \mid \sum_{i=2}^{n} x_i^2 < (\psi(x_1))^2, \ -\infty < x_1 < \infty \right\},
\]

where \( \psi \) is a bounded function with bounded derivative. We estimate the growth of the solutions in terms of \( \int_0^1 (1/\psi(r)) \, dr \). For the special case in which \( \varphi(\xi) = a\xi \ln \xi + C \), the solutions \( u(x_1, x_2, \ldots, x_n) \) grow as \( \left( \int_0^1 (1/\psi(r)) \, dr \right)^{N} \), where \( N \) is any given number and \( a = a(N) \).

KEY WORDS: elliptic inequalities, maximum principle, growth of solutions.

Introduction

We consider an elliptic operator of the form

\[
L = \sum_{i,k=1}^{m} a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k},
\]  

where \( a_{ik} = a_{ki} \), in an unbounded domain \( G \subset \mathbb{R}^m \) under the strict ellipticity condition

\[
\lambda |\xi|^2 \leq \sum_{i,k=1}^{m} a_{ik}(x)\xi_i \xi_k \leq \lambda^{-1} |\xi|^2 \quad \forall \xi \in \mathbb{R}^m, \ x \in G.
\]

In this note we study the behavior of the solutions of the differential inequality

\[
|Lu| \leq \varphi(|\nabla u|),
\]

\[
u_{|\partial G} = 0
\]

in the unbounded domain

\[
G = \left\{ x \in \mathbb{R}^m \mid \sum_{i=2}^{n} x_i^2 < (\psi(x_1))^2, \ -\infty < x_1 < \infty \right\}.
\]

Here \( \psi(t) \) is a continuously differentiable function and \( \varphi(\xi) \) is continuous, positive for \( \xi > 0 \), and monotone increases as \( \xi \to \infty \).

First, we study the special case in which \( \varphi(\xi) = a\xi \ln \xi \) (Theorem 1). In this case, we prove that the solution of (0.3), (0.4) grows as a power-law function. In addition, this special case clarifies the proof of Theorem 2, in which \( \varphi(\xi) \) is any function growing more slowly than \( a\xi \ln \xi \).
§ 1. The special case

First, we state some auxiliary lemmas.

**Lemma 1** (the maximum principle). Suppose that $D \subset \mathbb{R}^m$ is a bounded open set, $u(x), v(x) \in C^2(D) \cap C(\overline{D})$, and the following inequalities hold:

$$|Lu| \leq \varphi(|\text{grad } u|), \quad |Lv| < -\varphi(|\text{grad } v|),$$

where $\varphi(\xi)$ is defined and nonnegative for $\xi \geq 0$ and

$$\inf_{x \in D} \varphi(|\text{grad } v(x)|) > 0.$$

Moreover, let $u|_{\partial D} \leq v|_{\partial D}$; then $u \leq v$ in $D$.

**Proof.** Suppose that the assertion fails. Then there exists a point $x_0 \in D$ such that

$$u(x_0) - v(x_0) = \max_{x \in D} (u(x) - v(x)).$$

Therefore, $\text{grad } u(x_0) - \text{grad } v(x_0) = 0$. Set

$$\xi_0 = |\text{grad } u(x_0)| = |\text{grad } v(x_0)|.$$

Then we have

$$Lu|_{x=x_0} \geq -\varphi(\xi_0), \quad -Lv|_{x=x_0} > \varphi(\xi_0).$$

Hence, $L(u - v)|_{x=x_0} > 0$, which is a contradiction. □

**Lemma 2.** Let $f(r), 0 < r < \infty$, be a positive monotone increasing function, and let $\alpha$ and $\rho$ be numbers satisfying the inequalities $0 < \alpha < 1$ and $\rho > 0$. Furthermore, let the following inequality hold for sufficiently large positive integer $n$:

$$f((n + 1)\rho) \geq f(n\rho) + (f(n\rho))^\alpha.$$

Then for any $\varepsilon > 0$ and for sufficiently large $r$ we have

$$f(r) > r^{1/(1-\alpha)-\varepsilon}.$$

**Proof.** The proof can be found in [1]. □

**Lemma 3.** Let $x_0 \in \mathbb{R}^m$ be a given point. Then for any $s > 0$ and for any $x \neq x_0$ the following relations hold:

$$L \left( \frac{1}{|x - x_0|^s} \right) \geq \frac{s[(s + 2)\lambda - m\lambda^{-1}]}{|x - x_0|^{s+2}},$$

$$\text{grad } \frac{1}{|x - x_0|^s} = \frac{s}{|x - x_0|^{s+1}}.$$

**Proof.** The proof is by straightforward differentiation followed by an application of inequality (0.2). □

We introduce the following notation:

$$\Omega_{\psi} = \left\{ x \in \mathbb{R}^m \left| \sum_{i=1}^{m} x_i^2 < (\psi(x_1))^2, \ -\infty < x_1 < \infty, \ \psi(t) \in C(\mathbb{R}) \right. \right\},$$

$$Q_r^x = \left\{ y \in \mathbb{R}^m \left| |x - y| < r \right. \right\}, \quad S_r^x = \partial Q_r^x.$$