Uniqueness and Approximation Theorems for a Degenerate Operator-Differential Equation

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We consider the Cauchy problem

\[ \sum_{j=0}^{n} A_j u(j)(t) = 0, \quad t > 0, \quad (1) \]
\[ u(j)(0) = u_j, \quad j = 0, \ldots, n - 1. \quad (2) \]

Here the \( A_j \) are closed linear operators from a Banach space \( X \) into a Banach space \( Y \) with domains \( D_j \). Equations of the form (1) were studied by numerous authors under the assumption that the operator \( A_n \) is invertible. We do not assume that \( A_n \) or any other of the operators \( A_j \) are invertible. In that connection, Eq. (1) will be referred to as degenerate. The study of the general equation (1) in the present paper is partly inspired by the paper [1], where uniqueness and approximation issues were considered under different restrictions imposed on the resolvent of the characteristic pencil. Exponential estimates of the resolvents and the use of the Phragmen–Lindelöf principle are typical of these problems [2–4]. By a solution of the Cauchy problem (1), (2) we mean functions

\[ u(t) \in C^n((0, \infty), X) \cap C^{n-1}([0, \infty), X) \]

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satisfying (1), (2) and such that
\[ A_j u(t) \in C^j \{ (0, \infty), Y \} \cap C^{j-1} \{ [0, \infty), Y \}, \quad j = 1, \ldots, n. \]

For the initial data (2), consider the vector \( h_0 = \{ u_j \}_{j=0}^{n-1} \in X^n \). By \( X_0 \) we denote the closure (in \( X^n \)) of the set of all vectors \( h_0 \) for which the Cauchy problem (1), (2) is solvable. The subspace \( X_0 \) need not coincide with \( X^n \) and even may be trivial; it is the least closed subspace of \( X^n \) containing the vectors \( h_t = \{ u_j(t) \}_{j=0}^{n-1} \) for all solutions \( u(t) \) of Eq. (1) and all \( t \geq 0 \). The subspace \( X_0 \) will be called the \( n \)-fold subspace of solutions of the Cauchy problem (1), (2) or simply the solution subspace.

To Eq. (1) we assign the operator pencil
\[ L(\lambda) = \sum_{j=0}^{n} \lambda^j A_j \]
defined on \( D = \bigcap_{j=0}^{n} D_j \). It is assumed that \( D \neq \{0\} \), since otherwise the Cauchy problem (1), (2) has only the trivial solution. For the numbers \( \lambda \) that are not eigenvalues, we consider the resolvent operators \( R(\lambda) = L^{-1}(\lambda) \) and \( R_j(\lambda) = R(\lambda)A_j \), \( j = 1, \ldots, n \). A point \( \lambda \) is said to be weakly regular for \( L(\lambda) \) if the resolvent operators \( R_j(\lambda) \) are everywhere defined on \( \tilde{D} = \bigcap_{j=1}^{n} D_j \). Let \( E = \tilde{D} \) be the Banach space with the norm \( \| x \|_E = \| x \|_X + \sum_{j=1}^{n} \| A_j x \|_Y \). On the set of weakly regular points, all \( R_j(\lambda) \) are holomorphic as operator functions ranging in \( [E, X] \) and simultaneously in \( [E, E] \).

§1. The graph norm corresponding to the operator \( A_j \) will be denoted by \( \| x \|_j = (\| x \|_X + \| A_j x \|_Y) \).

**Theorem 1.** Suppose that the right half-plane contains a ray \( I = \{ \lambda(r) = \lambda_0 + re^{i\theta}, 0 \leq r < \infty \} \) (\( \Re \lambda_0 > 0, |\theta| < \pi/2 \)), on which there are no eigenvalues of \( L(\lambda) \), and let at least one of the two estimates
\[ \left\| R(\lambda) \sum_{j=1}^{n} A_j v_j \right\|_X \leq Ce^{\theta \Re \lambda} \sum_{j=1}^{n} \| v_j \|_j, \quad \lambda \in I, \tag{3} \]
\[ \sum_{k=1}^{n} \left\| A_k R(\lambda) \sum_{j=1}^{n} A_j v_j \right\|_Y \leq Ce^{\theta \Re \lambda} \sum_{j=1}^{n} \| v_j \|_j, \quad \lambda \in I, \tag{4} \]
be satisfied on this ray for some \( C > 0 \) and \( \beta \geq 0 \) and for any \( v_j \in D_j \) for which the left-hand side is defined. Then the solution of the Cauchy problem (1), (2) is unique.

To prove the theorem, note that the Laplace transform
\[ \tilde{u}_t(\lambda) = \int_0^t e^{-\lambda s} u(s) \, ds \]
of the cutoff of the solution \( u(t) \) to Eq. (1) with zero initial conditions (2) admits the following representation on \( I \):
\[ \tilde{u}_t(\lambda) = -e^{-\lambda t} R(\lambda) \sum_{j=1}^{n} A_j \sum_{k=0}^{j-1} \lambda^{k} u(j-1-k)(t), \quad \lambda \in I. \tag{5} \]

For fixed \( \gamma > \beta \) and any \( t > \gamma \), let us consider the entire vector functions \( z(\lambda) = e^{(t-\gamma)\lambda} \tilde{u}_t(\lambda) \) and \( z_k(\lambda) = e^{(t-\gamma)\lambda} A_k \tilde{u}_t(\lambda) \) \( (k = 1, \ldots, n) \). The ray \( I \) splits the half-plane \( \Re \lambda \geq \Re \lambda_0 \) into two angular domains, where we apply the Phragmen–Lindelöf principle for the function \( z(\lambda) \) under condition (3) or for the function \( z_k(\lambda) \) under condition (4). Next, by using the inversion formula for the Laplace transform of \( z(\lambda) \) or \( z_k(\lambda) \), we obtain \( u(t) \equiv 0 \) or \( A_k u(t) \equiv 0 \), whence, by (5), the assertion of the theorem follows. \( \square \)