A LOSS FUNCTION FOR ALPHA FACTOR ANALYSIS

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Most of the factor solutions can be got by minimizing a corresponding loss function. However, up to now, a loss function for the alpha factor analysis (AFA) has not been known. The present paper establishes such a loss function for the AFA. Some analogies to the maximum likelihood factor analysis are discussed.

Key words: alpha factor analysis, maximum likelihood factor analysis, loss function.

Whereas it is well known that the unweighted least squares (ULS), the generalized least squares (GLS) and the maximum likelihood (ML) solutions of factor analysis can be got by minimizing a corresponding loss function (see e.g. Jöreskog, 1977) hitherto an adequate loss function for the alpha factor analysis (AFA) of Kaiser and Caffrey (1965) has not been known. It is the purpose of the present paper to show that the function \( K \),

\[
K = \text{tr} \left[ (C - S^2)^{-1} (A - S^2) \right] - \ln |(C - S^2)^{-1} (A - S^2)| - n, \tag{1}
\]

is such a loss function, i.e. minimizing \( K \) leads to the AFA solution. By \( A \) we denote the observed \( n \times n \) covariance matrix.

\[
S^2 = \text{diag} \ A \tag{2}
\]

is the diagonal matrix of the observed variances.

\[
C = FF' + U^2, \tag{3}
\]

with \( F \) as the \( n \times r \) matrix of the loadings and \( U^2 \) as the \( n \times n \) diagonal matrix of the uniquenesses, is the estimated covariance matrix. We shall write \( H^2 \) for the difference

\[
H^2 = S^2 - U^2. \tag{4}
\]

(The derivation will show \( H^2 \) to be equal to the diagonal matrix \( \text{diag} \ FF' \) of the communalities.) In the following we shall use the abbreviations \( a, c, \)

\[
a = A - S^2 = A - H^2 - U^2, \tag{5}
\]

\[
c = C - S^2 = FF' + U^2 - S^2 = FF' - H^2. \tag{6}
\]

Because of (2)

\[
\text{diag} \ a = 0 \tag{7}
\]

holds. The equation (1) can then be rewritten as

\[
K = \text{tr} (c^{-1} a) - \ln |c^{-1} a| - n. \tag{8}
\]

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In order to show that $K$ is a loss function we have to minimize it with respect to $F$ and $U^2$ on which it depends via $c$. Using matrix derivatives (see e.g. Jöreskog, 1977) we get from (8)

$$dK = \text{tr} \left[ (c^{-1} - c^{-1}ac^{-1})dc \right].$$  \hspace{1cm} (9)

**Partial Derivative with Respect to $F$, Given $U^2$**

In this case from (6) there follows

$$dc = dF \cdot F' + FdF'.$$  \hspace{1cm} (10)

Inserting this into (9) gives

$$dK = 2 \text{tr} \left[ (c^{-1} - c^{-1}ac^{-1})FdF' \right]$$  \hspace{1cm} (11)

from which we get

$$\frac{\partial K}{\partial F} = 2(c^{-1} - c^{-1}ac^{-1})F.$$  \hspace{1cm} (12)

This leads to the equation

$$ac^{-1}F = F$$  \hspace{1cm} (13)

for the conditional minimum with given $U^2$. The inverse $c^{-1}$ can be expressed as

$$c^{-1} = H^{-2}F(\Lambda - I_r)^{-1}F'H^{-2} - H^{-2}$$  \hspace{1cm} (14)

where $\Lambda$ is at the moment only an abbreviation of $F'H^{-2}F$,

$$\Lambda = F'H^{-2}F,$$  \hspace{1cm} (15)

and $I_r$ is the $r \times r$ identity matrix. The equation (14) can easily be verified by pre- or post-multiplying its right-hand side by $c$ in the form $FF' - H^2$ (eq. (6)). Inserting of (14) into (13) gives after slight simplification

$$aH^{-2}F = F(\Lambda - I_r).$$  \hspace{1cm} (16)

Because of (5) this is equivalent to

$$(A - U^2)H^{-2}F = FA.$$  \hspace{1cm} (17)

In order to remove the indeterminateness with respect to orthogonal transformations it is usual to select the so-called canonical solution by restricting $\Lambda$ to being diagonal. Then, according to (17), the elements $\lambda_1, \lambda_2, \ldots, \lambda_r$ of $\Lambda$ are the “retained” eigenvalues of $(A - U^2)H^{-2}$ whereby “retained” means related to factors. The columns of $F$ are the associated eigenvectors. Because of (16) $\lambda_1 - 1, \ldots, \lambda_r - 1$ are the corresponding eigenvalues of $aH^{-2}$, the eigenvectors being the same. Now we multiply (3) in form $C - U^2 = FF'$ by $H^{-2}F$ from the right-hand side and apply (15):

$$(C - U^2)H^{-2}F = FA.$$  \hspace{1cm} (18)

Subtracting $F$ on both sides gives

$$cH^{-2}F = F(\Lambda - I_r).$$  \hspace{1cm} (19)

As $C - U^2$ has rank $r$ from (18), (19) it follows that

$$(\lambda_1 - 1)^{-1}, \ldots, (\lambda_r - 1)^{-1}; -1, \ldots, -1$$  \hspace{1cm} (20)

are the eigenvalues of $H^2c^{-1}$. Denoting the “rejected” eigenvalues of $(A - U^2)H^{-2}$ by