Integral Representation and Stabilization of the Solution to the Cauchy Problem for an Equation With Two Noncommuting Operators

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ABSTRACT. We obtain an integral representation for the solution to the Cauchy problem

\[ \frac{dv}{dt} = B_1^2 v + \frac{1}{2} b(t)(B_2 B_1 + B_1 B_2)v + c(t)B_2^2 v, \quad v(0) = v_0, \]

where the operators \( B_1 \) and \( B_2 \) are the infinitesimal generators of strongly continuous groups and \( B_1 B_2 - B_2 B_1 = kI, \ k \neq 0 \). For the case in which \( k = ik_1, \ k_1 \in \mathbb{R} \), it is proved that the solution tends to zero as \( t \to +\infty \).

Let \( B \) be the infinitesimal generator of a strongly continuous group \( T(t) \) in a Banach space \( E \). Then the semigroup \( U(t) \) generated in \( E \) by the Cauchy problem for the differential equation of the form

\[ \frac{dv}{dt} = B^2 v \]

admits the integral representation

\[ U(t) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp(-s^2/4t)T(s) \, ds, \quad t > 0, \]

and \( U(0) = 1 \).

Similar integral representations are known for some more general abstract first-order differential equations \([1, 2]\), in particular, for the equation

\[ \frac{dv}{dt} = \sum_{i=1}^{n} B_i^2 v, \]

where \( B_i \) are commuting infinitesimal generators of strongly continuous groups \( T_i(t) \). These integral representations differ in their kernels, which are fundamental solutions to the corresponding parabolic differential equations.

In the present paper we construct an integral representation of solutions to the equation

\[ \frac{dv}{dt} = B_1^2 v + \frac{1}{2} b(t)(B_1 B_2 + B_2 B_1)v + c(t)B_2^2 v, \]

where the operators \( B_i \) \((i = 1, 2)\) are the infinitesimal generators of strongly continuous groups \( T_i(t) \) \((T(0) = 1)\) and satisfy

\[ [B_1, B_2] \equiv B_1 B_2 - B_2 B_1 = kI \]

with some \( k \neq 0 \).

Differential equations with concrete noncommuting operators \( B_1 \) and \( B_2 \) satisfying these properties were considered in numerous papers (see \([3]\) and the bibliography therein). In these papers Lie algebra techniques were essentially used, and the obtained formulas for the solutions admitted no integral representation.

Furthermore, in the present paper we give sufficient conditions for the solution to the Cauchy problem for Eq. (2) to possess the semigroup property.

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Lemma 1. Suppose that \([B_1, B_2] = k1\) and

\[
Q_i(t) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp(-s^2/4t)T_i(s) \, ds, \quad i = 1, 2.
\]

Then the following relations hold:

(a) \([B_1^2, B_2] = 2kB_1\),
(b) \([B_2^2, B_1] = 4kB_2B_1 + 2k^21\),
(c) \([T_1(t), B_2] = kT_1(t)\),
(d) \([T_1(t), B_2^2] = (2kB_2 + k^2t^21)T_1(t)\),
(e) \([Q_1(t), B_2] = 2kB_1Q_1(t)\),
(f) \([Q_2(t), B_1] = 4k(t^2B_1^2 + B_2B_1 + \frac{k}{2}1)Q_1(t)\),
(g) \(T_1(t)T_2(t) = \exp(k\tau\tau)T_2(\tau)T_1(t)\).

Proof. Relations (a) and (b) can be proved by straightforward computation. To prove (c), note that the functions

\[
U_1(t) = \langle \_{B_1}T_2(t)u_0, \quad U_2(t) = T_2(t)\langle \_{B_1}u_0 + kT_2(t)u_0
\]

are solutions to the Cauchy problem

\[
\frac{du}{dt} = B_2u + kT_2(t)u_0, \quad u(0) = B_1u_0,
\]

where

\[
B_1u_0 \in D(B_2), \quad B_2u_0 \in D(B_2).
\]

By the uniqueness theorem [4, p. 641], we have \(U_1 = U_2\), which proves (c). Relation (d) is an immediate consequence of (c). To prove property (e), let us consider the function \(Q_1(t)B_2u_0\) and apply relation (c). We obtain

\[
Q_1(t)B_2u_0 = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp(-s^2/4t)T_1(s)B_2u_0 \, ds
\]

\[
= B_2Q_1(t)u_0 + k(4\pi t)^{-1/2} \int_{-\infty}^{\infty} s \exp(-s^2/4t)T_1(s)u_0 \, ds
\]

\[
= B_2Q_1(t)u_0 + kv_1(t),
\]

where

\[
v_1(t) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} s \exp(-s^2/4t)T_1(s)u_0 \, ds.
\]

Straightforward verification shows that the functions \(v_1(t)\) and \(v_2(t) = 2kB_1Q_1(t)u_0\) are solutions to the Cauchy problem

\[
\frac{du}{dt} = B_2u + 2B_1Q_1(t)u_0, \quad u(0) = 0,
\]

where

\[
u_1(t) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} s \exp(-s^2/4t)T_1(s)u_0 \, ds.
\]

By the uniqueness theorem, we have \(v_1 = v_2\), which proves (e).

Equation (f) follows from (e). Finally, to prove (g), we note that the functions \(T_1(t)T_2(\tau)u_0\) and \(\exp(k\tau\tau)T_2(\tau)T_1(t)u_0\) are solutions to the Cauchy problem \(du/dt = B_1u, \quad u(0) = T_2(\tau)u_0\). This completes the proof of the lemma.

In what follows, we consider the solution to (2) with the initial condition

\[
v(0) = v_0.
\]