Distance Matrices for Points on a Line, on a Circle, and at the Vertices of an $n$-Dimensional Cube

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ABSTRACT. For $n$ points $A_i$, $i = 1, 2, \ldots, n$, in Euclidean space $\mathbb{R}^m$, the distance matrix is defined as a matrix of the form $D = (D_{i,j})_{i,j=1}^{n}$, where the $D_{i,j}$ are the distances between the points $A_i$ and $A_j$. Two configurations of points $A_i$, $i = 1, 2, \ldots, n$, are considered. These are the configurations of points all lying on a circle or on a line and of points at the vertices of an $m$-dimensional cube. In the first case, the inverse matrix is obtained in explicit form. In the second case, it is shown that the complete set of eigenvectors is composed of the columns of the Hadamard matrix of appropriate order. Using the fact that distance matrices in Euclidean space are nondegenerate, several inequalities are derived for solving the system of linear equations whose matrix is a given distance matrix.

An original method of interpolating a function with given values at nonequidistant nodes was proposed in [1]. The method in question is based on solving equations of the form

$$\sum_{j=1}^{n} D_{i,j} p_j = D_{i,0}, \quad i = 1, 2, \ldots, n. \quad (1)$$

Here and below $D_{i,j}$ denotes the distance in Euclidean space $\mathbb{R}^m$ between the $i$th and $j$th points in a system of $n$ different points $A_i$, $i = 1, 2, \ldots, n$, with known values $f_i$ of the function $f$ at these points ($D_{i,i}$ are, naturally, assumed equal to zero), $D_{i,0}$ denotes the distance between the points $A_i$ and $A_0$ at which the interpolated value is to be determined, and the $p_i$ are unknown weight coefficients. Then, the interpolated value of the function is given by

$$f_0 = \frac{\sum_{i=1}^{n} p_i f_i}{\sum_{i=1}^{n} p_i}. \quad (2)$$

The distance matrix of the points $A_i$, $i = 1, 2, \ldots, n$, is the matrix of the set of equations (1). Naturally, the question of the nondegeneracy of this matrix arises (although the nondegeneracy condition is not a necessary one for some right sides).

The following inequality has been proved in [2] for $n$-point distance matrices in $\mathbb{R}^m$:

$$(-1)^{n-1} \det \left( \left( r^2 + D_{i,j}^2 \delta \right)_{i,j=1}^{n} \right) > 0, \quad (3)$$

which is valid for all real $r$ and $\tau$, $\delta$ in the ranges $0 < r \leq 1$ and $0 < \delta < 1$. For $r = 0$, $\tau = 1$, $\delta = 1/2$, inequality (3) implies

$$(-1)^{n-1} \det(D) > 0, \quad (4)$$

On the basis of this, it was proved in [3] that the distance matrix for $n$ distinct points in $\mathbb{R}^m$ has $n-1$ negative eigenvalues and one positive eigenvalue, which is, obviously, equal to the sum of the absolute values of the other eigenvalues.

Let us consider a configuration all of whose points $A_i$, $i = 1, 2, \ldots, n$, are distinct and are located on a circle or a line. It can be assumed that the enumeration of the points agrees with their order on the circle or the line when moving along a particular direction. In any case the points can always be enumerated in such a way that this condition is satisfied. Clearly, the renumbering of the points does not change the
value of the determinant of the distance matrix, since this leads only to the corresponding interchange of its rows and columns.

We shall introduce the following notation:

\[
i \oplus 1 = \begin{cases} 
  i + 1, & 1 \leq i \leq n - 1, \\
  1, & i = n,
\end{cases}
\]

\[
i \ominus 1 = \begin{cases} 
  i - 1, & 2 \leq i \leq n, \\
  n, & i = 1,
\end{cases}
\]

\[
\delta_{i,j} = \begin{cases} 
  1, & i = j, \\
  0, & i \neq j.
\end{cases}
\]

Then the following theorem holds.

**Theorem 1.** If the points \( A_i, i = 1, 2, \ldots, n \), are distinct and located on a circle or a line, then the distance matrix \( D = (D_{i,j})_{i,j=1,\ldots,n+1} \) of these points is nondegenerate, and its inverse, the matrix \( Q \), is of the form

\[
Q = (Q_{i,j})_{i,j=1,\ldots,n}, \quad Q_{i,j} = \frac{D_{i \oplus 1, i} \delta_{i \ominus 1, j} + D_{i \ominus 1, i} \delta_{i \oplus 1, j} - D_{i \ominus 1, i} \delta_{i, j}}{2D_{i \ominus 1, i} D_{i \ominus 1, i}}.
\]

**Proof.** Let \( T \) denote the product of the matrices \( Q \) and \( D \). Its \((i, j)\)th entry is determined by

\[
T_{i,k} = \sum_{j=1}^{n} Q_{i,j} D_{j,k} = \frac{D_{i \oplus 1, i} D_{i \ominus 1, k} + D_{i \ominus 1, i} D_{i \ominus 1, k} - D_{i \ominus 1, i} D_{i, k}}{2D_{i \ominus 1, i} D_{i \ominus 1, i}}
\]

Consider the case in which the points are located on a circle. If \( k \neq i, \ i \ominus 1, \ i \oplus 1 \), then the points \( A_i \ominus 1, A_i, A_i \oplus 1, A_k \) constitute the vertices of a convex quadrilateral inscribed in a circle. Then, by Ptolemy's theorem, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals, i.e., to the numerator in (7) and hence \( T_{i,k} = 0 \). It is readily seen from (5) that \( T_{i, i \ominus 1} = T_{i, i \oplus 1} = 0, T_{i,i} = 1 \). Thus, \( T \) is the unit matrix, and \( Q \) is the inverse of \( D \).

If the points are located on a circle, then the assertion of the theorem follows immediately from (7). \( \square \)

It follows from (6) that the inverse matrix \( Q \) has, so to speak, a cyclically three-diagonal form. Only the entries in the principal diagonal, in the superdiagonal, and in the subdiagonal, as well as the entries in the top right and bottom left, are nonzero.

It should be noted that the paper [3] deals with the cases in which all the points lie on a single line and constitute the vertices of a regular \( n \)-gon.

**Corollary 1.** Under the conditions of Theorem 1, the solution of the system of equations (1) for any point \( A_0 \) exists and is nonnegative, i.e., \( p_i \geq 0, i = 1, 2, \ldots, n \).

Indeed, the solution of (1) is determined by

\[
p_i = \sum_{j=1}^{n} Q_{i,j} D_{j,0} = \frac{D_{i \oplus 1, i} D_{i \ominus 1, 0} + D_{i \ominus 1, i} D_{i \ominus 1, 0} - D_{i \ominus 1, i} D_{i, 0}}{2D_{i \ominus 1, i} D_{i \ominus 1, i}}, \quad i = 1, 2, \ldots, n.
\]

But in view of the inequality proved in [4] as a generalization of Ptolemy's theorem, the numerator in (8) is greater than or equal to zero; thus, \( p_i \geq 0, i = 1, 2, \ldots, n \).

\[\text{For any four distinct points } A_j, A_k, A_l, A_m \text{ in Euclidean space the inequality } D_{j,k} D_{l,m} + D_{j,m} D_{k,l} \geq D_{j,l} D_{k,m} \text{ holds; the equality is valid only if these points lie on a circle or on a line and follow one another in the order } j, k, l, m.\]