Kolmogorov $n$-Width of Some Finite-Dimensional Sets in a Mixed Measure

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Calculation of upper and lower bounds for $n$-widths of some functional classes often reduces to estimates for $n$-widths of finite-dimensional sets. For example, in [1] a lower bound for the Kolmogorov $n$-width $d_N(H_p^s(T^n), L_q)$ of the class $H_p^s(T^n)$ of periodic functions with dominating mixed derivative in the space $L_q$ for $1 < p < q < 2$ is calculated. The calculation is reduced to that of a lower bound for the $n$-width $d_N(B_1,\infty, \ell_2^n, \ell_1^m)$ of the finite-dimensional set $B_1,\infty$ in the mixed norm $\ell_2^n \ell_1^m$ by means of the Littlewood–Paley theorem, an inequality due to V. N. Temlyakov and the Martsinkievich–Sigmund discretization theorem. The following theorem gives the asymptotics for this $n$-width.

Theorem [1]. Let $1 < q < \infty$, $N \leq nm/2$. Then

$$d_N(B_1,\infty, \ell_2^n, \ell_1^m) \asymp n^{1/q}.$$

The upper bounds are trivial for all values of $q$. The proof of the lower bound for $q = \infty$ coincides with that from [1]. In my talk at B. S. Kashin's seminar, I stated this problem for the case $q = 1$. This case requires new ideas and it may prove useful for calculation of the linear widths for the Hölder–Nikolsky classes and Besov classes. Using probability methods, A. D. Isaac succeeded in obtaining the following nontrivial estimate.

Theorem (A. D. Isaac [2]). Let $N \leq nm/2$. Then

$$m^{\frac{\log \log m}{\log m}} \ll d_N(B_1,\infty, \ell_2^n, \ell_1^m) \ll m.$$

However a complete solution for the stated problem was not found.

In the present paper we determine the asymptotics for the Kolmogorov $n$-width $d_N(V_{k,\infty}^p, \ell_1^m)$ of the generalization $V_{k,\infty}^p$ of the set $B_1,\infty$ in the mixed norm $\ell_1^m$ for $p = 2, 1 < q \leq \infty$ and $1 < p \leq \infty$. 


\[ \min\{q, 2\}, \quad 1 < q \leq \infty. \]

The sets \( V_{k, \infty}^{n,m} \) appear, for example, in the lower bounds for the widths of the intersection of the Hölder–Nikolsky classes \( H_p^r(T^n) \) of periodic functions in several variables. Here \( B_p^{n,m} \) is the unit ball in \( \ell_p^{n,m} \) in the norm
\[
\|x\|_{\ell_p^{n,m}} = \left( \sum_{s=1}^{m} \left( \sum_{k \in \Delta_s} |x_k|^p \right)^{q/p} \right)^{1/q}, \quad 1 \leq p, q \leq \infty, \quad x \in \mathbb{R}^{n,m},
\]
where \( \Delta_s = \{ k \in \mathbb{N} \mid (s-1)n < k \leq sn \}, \quad s = 1, \ldots, m. \)

Denote by \( V_k^n \) the convex hull in \( \mathbb{R}^n \) of the points having \( k \) coordinates equal to \( \pm 1 \) and all other coordinates equal to zero. These sets are the usual tool used for finding lower bounds of the widths of finite-dimensional sets and functional classes. Then \( B_1, \infty = V_1^n \times \cdots \times V_1^n, \quad V_1, \infty = V_1^n \times \cdots \times V_1^n. \)

The following lemma will be useful in establishing the lower bounds for the widths of finite-dimensional sets in the mixed norm.

**Lemma.** Let \( T = \{ t = \bigcup_{s=1}^{m} t_s \mid t_s \subset \Delta_s, \quad \text{card} t_s = k, \quad s = 1, \ldots, m \} \); \( 0 \leq x_i \leq 1, \quad i = 1, \ldots, mn, \)
\( \sum_{i=1}^{mn} x_i \leq n; \quad 0 < q < \infty. \) Then
\[
I = \left( \frac{1}{|T|} \sum_{t \in T} \left( \sum_{i \in t} x_i \right)^{q/p} \right)^{1/q} \leq C_k,
\]
where the constant \( C \) depends on \( q \) and does not depend of \( k, n, m \) and on the set \( x_i \).

This lemma generalizes Lemma 2 from [1]. Its proof is very close to that of Lemma 2 from [1], but it contains some new features. For this reason we present a complete proof.

**Proof.** The mean inequality yields
\[
\left( \frac{1}{|T|} \sum_{t \in T} y_t^q \right)^{1/p} \leq \left( \frac{1}{|T|} \sum_{t \in T} y_t^q \right)^{1/q}
\]
for \( 0 < p \leq q \leq \infty, \quad y_t \geq 0, \quad t \in T. \) It is then sufficient to establish the upper bound for \( q \in \mathbb{N}. \) Raising to the power \( q \), we obtain
\[
I^q = \frac{1}{|T|} \sum_{t \in T} \sum_{a \in A} a_1^{l_1} \cdots a_{km}^{l_{km}} x_{l_1}^{a_1} \cdots x_{l_{km}}^{a_{km}} \leq \frac{1}{|T|} \sum_{t \in T} \sum_{a \in A} x_{l_1}^{a_1} \cdots x_{l_{km}}^{a_{km}},
\]
where \( A = \{ a \in \mathbb{Z}_+^{km} \mid \sum_{i=1}^{km} a_i = q \}, \quad t = \bigcup_{i=1}^{km} l_i, \quad l_i < l_j \) for \( i < j \); (in these formulas we set \( 0^0 = 1 \)).

Introduce an equivalence relation on \( A \) in the following way: two vectors belong to one and the same coset if each of them may be obtained from the other one by a transposition of coordinates. The number of cosets then does not exceed \( q^q. \) In order to estimate \( I \), it is then sufficient to estimate
\[
I_a^q = \frac{1}{|T|} \sum_{t \in T} \sum_{a \sim a'} x_{l_1}^{a_1} \cdots x_{l_{km}}^{a_{km}} = \frac{1}{|T|} \sum_{t \in T} \sum_{l_i, l_j} x_{l_1}^{a_1} \cdots x_{l_r}^{a_r},
\]
for a fixed vector \( a = (a_1, \ldots, a_r, 0, \ldots, 0) \in \mathbb{R}^{mk} \) with \( r \) nonzero coordinates \( (r \leq q) \). Since the number of terms in the sum over \( a \sim a \) does not exceed the number of arrangements of \( km \) elements on \( r \) places, i.e., the number \( A_{km}^r = (km)! / (kn-r)! \ll (km)^r \), and the sum over \( T \) is averaged over the number of terms, we get
\[
I_a^q \ll \frac{(km)^r}{A_{km}^r} \sum_{l_1, \ldots, l_r} x_{l_1}^{a_1} \cdots x_{l_r}^{a_r} \ll \frac{1}{|T|} \sum_{l_1, \ldots, l_r} \sum_{l_1, \ldots, l_r} x_{l_1}^{a_1} \cdots x_{l_r}^{a_r}
\]
\[
\leq \left( \frac{k}{n} \right)^r \left( \sum_{i=1}^{mn} x_i^{a_1} \right) \cdots \left( \sum_{i=1}^{mn} x_i^{a_r} \right). \]

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