On Continuous Extension of Locally Homeomorphic Simplicial Maps of $\mathbb{R}^2$ into Itself by $\sigma$-Processes

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ABSTRACT. It is shown that the Klein bottle with two points removed can be embedded in the compactification of $\mathbb{R}^2$ by a finite tree.

§1. Statement of the result

The note [1], which was devoted to a conjecture of Vitushkin concerning the Jacobian problem, contains the construction of a covering over $\mathbb{R}^2$, which in my opinion, may be of interest in the topological approach to this problem. Recall that this covering is a surface $M$ that consists of one open two-dimensional cell and three open one-dimensional cells $l_+, l_-, l$ and a proper map $F : M \to \mathbb{R}^2$ locally homeomorphic on the two-dimensional cell; in suitable local coordinates, the map $F$ can be written as $(x, y) \mapsto (x, y^2)$ near each point in $l_+ \cup l_-$ and as $(x, y) \mapsto (x, y)$ near each point of the one-dimensional cell $l$; and it takes each one-dimensional cell to a curve in $\mathbb{R}^2$ that moves away to infinity along a certain asymptotic direction at the endpoints of the cell. This article is a supplement to the author's work [1]; namely, we shall prove a number of additional properties of the above covering.

It is known that by attaching a finite number of Riemannian spheres to $\mathbb{C}^2$, we can complete it to a compact manifold such that a given polynomial map of $\mathbb{C}^2$ into itself extends to a continuous map of this manifold into $\mathbb{C}P^1 \times \mathbb{C}P^1$. If these polynomials have real coefficients, we can consider a similar construction, which consists of a closed surface obtained by completing $\mathbb{R}^2$ by a finite number of circles and a continuous map of this surface to $\mathbb{R}P^1 \times \mathbb{R}P^1$. This system of circles results from a succession of $\sigma$-processes (blowups), that is, of attaching circles at ambiguous points of the map.

It should be noted that this system of circles constitutes a one-dimensional tree. The first of them is the circle at infinity in $\mathbb{R}P^2$; at each ambiguous point of the map on this circle a $\sigma$-process is performed; then the process is repeated at any ambiguous points that may appear on the new circles obtained at an earlier stage. The construction is terminated as soon as there are no ambiguous points left. So the map is continuously extended to the tree thus obtained. It is easy to see that any two circles of the resulting system either are disjoint or have a single common point. The common point of two circles is called a branch point of the tree. A circle of the tree is called an end circle if it contains only one branch point. An end circle is said to be a distinguished circle if the circle intersecting with it appears at an earlier stage of the construction than this circle itself.

The main result of this paper is the following statement.

Proposition 1. The surface $M$ can be embedded in a closed surface $\widetilde{M}$ consisting of one two-dimensional cell and a finite set of circles that form a tree; this embedding has the following properties:

(a) the two-dimensional cell of $M$ is mapped on the two-dimensional cell of $\widetilde{M}$, while the one-dimensional cells of $M$ are mapped into the distinguished circles of the tree in $\widetilde{M}$ so that each of them covers the entire corresponding distinguished circle except for one point, the branch point of the tree on this circle;

(b) the map $F$ is continuously extended to the entire surface $\widetilde{M}$ so that the extension acts on the circles of the tree in the following manner:

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three of the distinguished circles of the tree with their branch points removed are mapped into the finite part of $\mathbb{RP}^1 \times \mathbb{RP}^1$, i.e., into $\mathbb{R}^2$; the branch points are taken to the point $(\infty, \infty)$;
(ii) each of the remaining distinguished circles is homeomorphically mapped onto one of the circles \( \{x = \infty\}, \{y = \infty\} \) in $\mathbb{R}^2 \times \mathbb{R}^2$, while all the nondistinguished circles collapse to $(\infty, \infty)$.

The proof of this statement (see §4 below) is obtained by means of a general construction that ascertains whether a given open surface can be embedded in a closed surface of the particular form specified above.

§2. Normal form of a surface

It is known that any closed surface (i.e., a two-dimensional connected compact manifold without boundary) can be represented in the following way. Fix a unit circle in the plane and mark off equally spaced points $P_1, \ldots, P_{2n}$ on its boundary. Consider the regular polygon with vertices $P_1, \ldots, P_{2n}$. The edges of the polygon are divided into pairs and the edges of each pair are pasted to each other by the identification homeomorphism. In so doing, all the vertices $P_1, \ldots, P_{2n}$ are identified into one common point and the given surface is formed.

Orient the face of the polygon by taking, say, the counterclockwise direction on its boundary as positive. Denote the edges of each pair by the same letter if their orientations induced by the corresponding identification homeomorphism both agree or both disagree with the orientation of the polygon’s face; otherwise designate the two edges as, e.g., “w” and “w’”. Thereby, any closed surface can be defined by the sequence of the symbols corresponding to the edges of the regular polygon written in the ascending order of the indices of its vertices starting at $P_1$. This representation is called the normal form of the surface (for details, see [2, §3.2]).

For an open surface $\mathcal{M}$ that consists of one two-dimensional cell and a finite set of one-dimensional cells $l_1, \ldots, l_p$, we have the following representation.

Lemma 1. The open surface $\mathcal{M}$ can be obtained from a regular $2p$-gon $\mathcal{W}$ with deleted vertices by pairing its sides and pasting the intervals of each pair together; in so doing, each one-dimensional cell of $\mathcal{M}$ is obtained by pasting together the intervals of one of the pairs.

In [1] it is shown that the surface $M$ can be glued from a regular hexagon without vertices whose sides are copies of the curves $l_+, l_-, l$. Denote this hexagon by $W$; it realizes the polygonal representation of $M$ mentioned in Lemma 1. Fix a frame $(e_1, e_2)$ tangent to $M$ with origin at the point $P^* \in l$ so that $F(P^*) = 0$, the vector $e_1$ is directed along the one-dimensional cell $l$, and the vector $e_2$ is orthogonal to the first vector. If $M$ is thought of as a surface over $\mathbb{R}^2$ and the map $F$ as the projection of $M$ on $\mathbb{R}^2$, then the projection of the frame $(e_1, e_2)$ on the plane coincides with the standard frame $\mathbb{R}^2_x, y$. Choose the orientation of the two-dimensional cell of the surface $M$ in accord with the vector $e_1$. Then it is easy to see that the normal form of $M$ can be written as

$$W = abca^{-1}bc.$$

In this representation, the curves $l_+, l_-, l$ are obtained by pasting together the following pairs of oriented intervals:

$$l = (a, a^{-1}), \quad l_+ = (b, b), \quad l_- = (c, c).$$

Next, consider a closed surface $\overline{\mathcal{M}}$ consisting of $\mathbb{R}^2$ and the tree of circles $\Sigma_1, \ldots, \Sigma_q$. It is easy to see that the branch points divide the tree into an even number of one-dimensional cells $l_1, \ldots, l_{2(q-1)}$, so the representation of the surface $\overline{\mathcal{M}}$ specified above will be obtained from a regular polygon whose number of sides is a multiple of 4. As in Lemma 1, for closed surfaces we have the following description:

Lemma 2. The closed surface $\overline{\mathcal{M}}$ can be obtained from a regular $4q$-gon $\overline{\mathcal{W}}$ by pairing its sides and identifying the sides of each pair. In so doing,

(1) the interior of the polygon is associated with the two-dimensional cell of the surface $\overline{\mathcal{M}}$, that is, with $\mathbb{R}^2$;
(2) each of the cells $l_j$ is obtained by pasting together one of the pairs of sides of the polygon;
(3) each branch point is obtained by pasting together four vertices of the polygon.