A Few Remarks on $\zeta(3)$

Yu. V. Nesterenko

UDC 511.36

Abstract. A new proof of the irrationality of the number $\zeta(3)$ is proposed. A new decomposition of this number into a continued fraction is found. Recurrence relations are proved for some sequences of Meyer's $G$-functions that define a sequence of rational approximations to $\zeta(3)$ at the point 1.

In 1978 R. Apery proved the following assertion.

Theorem 1 [1]. The number

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is irrational, and for any constant $\mu > 13.4178202\ldots$ the inequality

$$\left| \zeta(3) - \frac{p}{q} \right| < q^{-\mu}$$

has only a finite number of solutions in integers $p$ and $q$.

The exact value of the constant $13.4178202\ldots$ is

$$1 + \frac{4\ln \beta + 3}{4\ln \beta - 3},$$

where $\beta = \sqrt{2} + 1$.

As is well known, for Riemann’s zeta-function

$$\zeta(s) = \sum_{\nu=1}^{\infty} \nu^{-s}$$

at even points, we have

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k},$$

where the $B_{2k} \in \mathbb{Q}$ are the Bernoulli numbers satisfying the recurrence relations

$$\sum_{r=0}^{n} \binom{n+1}{r} B_r = 0, \quad n = 1, 2, \ldots,$$

and the condition $B_0 = 1$. Thus, for example, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. These expressions prove the transcendence of the numbers $\zeta(2k)$. Before Apery’s result, nothing was known about the arithmetic properties of the zeta-function at odd points.

Apery’s proof is based on his discovery of a sequence of rational approximations to the number $\zeta(3)$. For example, this sequence can be constructed as the sequence of convergents of the continued fraction

$$\zeta(3) = \frac{6}{5} - \frac{1}{\frac{117}{535}} - \frac{64}{\frac{34n^3 + 51n^2 + 27n + 5}{\ldots}} \ldots$$  

(1)
that converges to \( \zeta(3) \); or by means of the following recurrence equation
\[
(n + 1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3u_{n-1} = 0
\]
that is satisfied by the numerators \( v_n \) and denominators \( u_n \) of the convergents with the initial conditions
\[
v_0 = 0, \quad v_1 = 6, \quad u_0 = 1, \quad u_1 = 5;
\]
or by the explicit representation of the sequences in question
\[
u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \cdot c_{n,k}, \quad n \geq 1,
\]
where
\[
c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3} \binom{n}{m}^{-1} \binom{n+m}{m}^{-1}
\]
The greatest difficulty in the proof is the verification of the fact that the sequences (3) satisfy the recurrence equation (2).

Apery himself only published a sketch of the proof. A detailed exposition of this proof can be found in [2-4].

During the 20 years that have passed since, nothing new in principle has been published in this connection. In various articles Apery's phenomenon was interpreted in a number of ways.

F. Beukers (1979, [5]) put forward another interpretation of Apery's rational approximations to \( \zeta(3) \). The following relation is valid:
\[
J = \int_0^1 \int_0^1 \frac{\ln(xy)}{1-xy} Q_n(x)Q_n(y) \, dx \, dy = 2(u_n \zeta(3) - v_n),
\]
where \( Q_n(x) = \binom{1}{1} \cdot \left( \frac{d}{dx} \right)^n \left( x^n(1-x)^n \right) \in \mathbb{Z}[x] \)
are Legendre polynomials and the \( u_n, v_n \) are Apery's numbers (3).

M. Hata (1990, [6]), modifying the integral \( J \) and, consequently, changing the rational approximations to \( \zeta(3) \), proved the assertion of Theorem 1 for any \( \mu > 8.8302837 \ldots \).

L. A. Gutnik (1979, [7]) established that for any rational number \( q, q \neq 0 \), there is an irrational number among the numbers
\[
3 \zeta(3) + q \cdot \zeta(2), \quad \zeta(2) + 2q \cdot \ln 2
\]
The polylogarithm
\[
L_r(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^r}, \quad r \geq 1,
\]
as well as the so-called Meyer G-functions (see [8]), play an important role in the proof. By means of these functions, the following explicit constructions were established (see [7]):
\[
G_n(z) = 2A_n(z) \cdot L_3(z^{-1}) + B_n(z) \cdot L_2(z^{-1}) + C_n(z), \quad H_n(z) = A_n(z) \cdot L_2(z^{-1}) + B_n(z) \cdot L_1(z^{-1}) + D_n(z),
\]
where \( A_n, B_n, C_n, D_n \) are polynomials in \( z \) with rational coefficients of degree at most \( n \) satisfying the conditions
\[
\text{ord}_{z=\infty} G_n(z) \geq n + 1, \quad \text{ord}_{z=\infty} H_n(z) \geq n + 1.
\]
This functional construction is used in [7] for \( z = -1 \). For the polynomials \( A_n, B_n, C_n, D_n \), there also exist explicit expressions which, in particular, imply that \( A_n(1) = u_n, C_n(1) = -2v_n \), and \( B_n(1) = 0 \).

F. Beukers (1981, [9]) proved the uniqueness of polynomials realizing the joint functional approximations indicated above.

Other problems involving joint functional approximations, which lead to the proof of the irrationality of \( \zeta(3) \), were treated by Sorokin in [10].

In what follows (§1), we propose a proof of Theorem 1, inspired by Gutnik's work (see [7]), and then (§2) prove Theorem 2, in which a new representation of \( \zeta(3) \) by a continued fraction is given.