Since for \( n = 1 \) the theorem is obvious, for \( n = 2m + 1, \ m \in \mathbb{N} \), we can take

\[
S_{2m+1}^\varphi(x) \equiv S_{2m}^\varphi(x).
\]

The proof of the theorem is complete. \( \square \)

I wish to thank S. B. Stechkin and V. N. Temlyakov for their valuable advice and interest in my work.

References


University of South Carolina

Translated by N. K. Kulman

Mathematical Notes, Vol. 59, No. 6, 1996

Minimum Deviation from Zero for the Chebyshev Mappings Corresponding to an Equilateral Triangle

I. V. Belyakov

Veselov [1] and Hoffman and Withers [2] suggested a construction that assigns a function (generalized cosine) \( h: \mathbb{R}^n \rightarrow \mathbb{R}^n \) to each affine Weyl group \( \widetilde{W} \). This function is \( \widetilde{W} \)-invariant (that is, \( h(g(x)) = h(x) \) for all \( g \in \widetilde{W} \)) and homeomorphic on a fundamental domain of \( \widetilde{W} \). Furthermore, \( h \) satisfies the polynomial transformation law

\[
h(kx) = T_k(h(x)), \quad k \in \mathbb{N}.
\]

The polynomial mappings \( T_k \) are known as the generalized Chebyshev polynomials corresponding to \( \widetilde{W} \) [2]. Their properties are in many respects similar to those of the classical Chebyshev polynomials, which are related to the group \( \widetilde{W}(A_1) \) generated by the reflections of the real axis in the endpoints of the interval \([0, \pi]\).

Here we deal with the group \( \widetilde{W}(A_2) \) generated by the reflections of the plane in the sides of an equilateral triangle. In this case, the generalized cosine \( h: \mathbb{R}^2 \rightarrow \mathbb{C} \) has the form

\[
h(u, v) = e^{2\pi i u} + e^{2\pi i v} + e^{-2\pi i (u+v)},
\]

and maps the plane into the deltoid (also known as the Steiner domain), which will be denoted by \( \Delta \) (see the accompanying figure).
The deltoid is bounded by the three branches of the hypocycloid specified by the parametric equation

\[ z = 2e^{2\pi i u} + e^{-4\pi i u}, \quad u \in [0, 1]. \]

The generalized Chebyshev polynomials \( A_n \) for which

\[ h(nu, nu) = A_n(h(u, v)) \]

define \( n^2 \)-foldings of the deltoid (see [3] for their properties). They can be expressed as polynomials in \( z = h(u, v) \) and \( \bar{z} \). The first three \( A_n \) are as follows:

\[ A_0(z) = 3, \quad A_1(z) = z, \quad A_2(z) = z^2 - 2\bar{z}. \]

Furthermore, the recurrence relation

\[ A_n = zA_{n-1} - \bar{z}A_{n-2} + A_{n-3}, \quad n \geq 3, \]

holds, whence it follows that the total degree of \( A_n \) in \((z, \bar{z})\) is equal to \( n \) and the leading term is \( z^n \).

In what follows, the subscript in the notation of a polynomial indicates the highest possible total degree of the polynomial; however, if there is a bar over the subscript, then the subscript indicates the highest possible degree in each of the variables.

**Theorem.** For polynomials \( P_{n-1}(z, \bar{z}) \) over \( \mathbb{C} \), let

\[ \mu = \inf_{P_{n-1}} \max_{z \in \Delta} |z^n + P_{n-1}(z, \bar{z})|. \]

Then \( \mu = 3 \), and the maximum is attained only for \( z^n + P_{n-1} \equiv A_n \).

**Proof.** 1) First, note that \( \max_{\Delta} |A_n + c| > \max_{\Delta} |A_n| \) if \( c = \text{const} \neq 0 \). Indeed, it follows from the inequality \( |A_n + c| \leq 3 \) for \( |A_n| = 3 \) that the vector \( c \) must belong to three half-planes whose intersection contains only zero.

2) Let us introduce the variables \( \xi = e^{2\pi i u}, \eta = e^{2\pi i v}, \) and \( \zeta = e^{2\pi i w} \), where it is assumed that \( u + v + w = 0 \). Then

\[ z = \xi + \eta + \zeta, \quad \bar{z} = \xi \eta + \zeta \eta + \zeta \zeta, \quad A_n = \xi^n + \eta^n + \zeta^n. \]

Next, \( z^n + P_{n-1}(z, \bar{z}) - A_n(z) = Q_{n-1}(\xi, \eta, \zeta) = Q_{n-1}(\zeta, \eta, \zeta) \). The substitution \( \zeta = 1/\xi \eta \) takes the monomial \( \xi^a \eta^b \zeta^c \) to \( \xi^{a-c} \eta^b \zeta^c \); hence, \( Q_{n-1}(\zeta, \eta, \zeta) \) is taken to the Laurent polynomial \( \tilde{Q}_{n-1}(\xi, \eta) \) with monomials of the form \( \xi^i \eta^j \), where \( |i| \leq n - 1, |j| \leq n - 1 \), and

\[ |i - j| \leq n - 1. \]

\[ (*) \]