Two-Point Boundary Value Problems in Relativistic Dynamics

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ABSTRACT. For gyro systems of relativistic type, we obtain solvability conditions for the two-point boundary value problem. We use the geodesic modeling method, in which the original problem is reduced to studying the existence of isotropic geodesic curves of the Kaluts-O. Klein Lorentz metric joining two fibers of a bundle over the configuration manifold of the system. As an example, we consider problems on the motion of charged test particles in an arbitrary electromagnetic field and in the outer Reissner-Nordstrom space-time in the field of a charged black hole and some external electromagnetic field.

We consider relativistic analogs of natural mechanical systems with gyro forces and derive solvability conditions for the two-point boundary value problem. We use the geodesic modeling method: on the total space of a principal bundle over the configuration manifold of the system a Lorentz metric of Kaluts-O. Klein type is introduced; the original problem is reduced to studying the existence of isotropic geodesic curves connecting two fibers of the bundle; the causal structure of the Lorentz manifold that serves as a model is studied. The results obtained are used to study the motion of particles in various gravitational and electromagnetic fields.

\$1\$. Suppose that \((M, g)\) is a Lorentz manifold, \(F\) is a closed 2-form, and \(u\) is a smooth function on \(M\). We assume that \(u(p) > 0\) for all \(p \in M\) and that the integrals of \(F\) over two-dimensional spheroids of \(M\) form a \(k\)-dimensional \(\mathbb{Z}\)-module \(P\), \(k \in \{0, 1\}\). The condition on \(k\) means that \(P = \emptyset\) for some \(\theta \in \mathbb{R}\). Moreover, we can assume that \(\theta\) is nonnegative, specifically, \(k = 0\) for \(\theta = 0\) and \(k = 1\) for \(\theta > 0\).

Consider some points \(p, q \in M\), the interval \(I = [0, 1]\), and the space \(\Omega_{pq}\) of piecewise smooth paths \(x: I \to M\) joining \(x(0) = p\) with \(x(1) = q\). In each homotopy class \(\Delta \in \pi_0(\Omega_{pq})\) we choose a base path \(x_\Delta\). Suppose that \(x \in \Omega_{pq}\) is some path, \(\Delta \in \pi_0(\Omega_{pq})\) is the homotopy class of \(x\), \(\delta \in \mathbb{R}\), and \(c: I^2 \to M\) is a piecewise smooth homotopy connecting \(x_\Delta\) and \(x\). Then we set \(\dot{x} = dx/d\tau\) and

\[
S_\delta(x, c) = \int_0^1 \left[ g(\dot{x}, \dot{x}) - \delta^2 u(x) \right] d\tau + \delta \int_c F.
\]

If \(c': I^2 \to M\) is another homotopy connecting the same paths, then we have the following inclusion: \(S_\delta(x, c') - S_\delta(x, c) \in \delta P\). Thus, the formula \(S_\delta(x) = S_\delta(x, c) + \delta P\) specifies a well-defined functional \(S_\delta: \Omega_{pq} \to \mathbb{R}/\delta P\).

We set \(G = \mathbb{R}\) and \(\alpha(r) = r\) if \(\theta = 0\) and \(r \in \mathbb{R}\), whereas \(G = U(1)\) and \(\alpha(r) = \exp(2\pi i r)\) if \(\theta \neq 0\) and \(r \in \mathbb{R}\). Further, we write \(\langle t \rangle = 1\) for \(t = 0\) and \(\langle t \rangle = t\) for \(t \in \mathbb{R} \setminus \{0\}\). The formula \(S(\delta, x) = \alpha(S_\delta(x)/(\delta\theta))\) specifies a functional \(S: \mathbb{R} \times \Omega_{pq} \to G\). The pairs \((\delta, x) \in \mathbb{R} \times \Omega_{pq}\) in which the path \(x\) is an extremal of the functional \(S_\delta\) will be called the extremals of the functional \(S(\delta, x)\).

A similar (multivalued) functional for a proper Riemannian metric was constructed in [1], and the existence of extremals of this functional was studied in [2]. The replacement of the proper Riemannian metric by a Lorentz metric results in a different (and substantially more complicated) statement of the problem. In fact, only some part of extremals of \(S\) are physically meaningful in applications in the Lorentz case. For example, the motion of a charged test particle in a gravitational and an electromagnetic fields is described by extremals \((\delta, x)\) such that \(\delta \neq 0\) and \(g(\dot{x}, \dot{x})/\delta^2 = -1\). Nevertheless, the methods and the results of [2] are widely used in the following.

If \((\delta, x)\) is an extremal of \(S\) and \(\delta > 0\), then the curve \(y: [0, \delta] \to M\) given by \(y(s) = x(s/\delta)\) is a solution of the equation


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where $\nabla$ is the connection of the Lorentz metric $g$ and $F^#$ is the tensor field of type $(1,1)$ given by $F(X, Y) = g(F^#(X), Y)$. Furthermore,

$$\left( \frac{\nabla}{ds} \right) \left( \frac{dy}{ds} \right) = F^# \left( \frac{dy}{ds} \right) - \text{grad} \, u(y),$$

(1)

where $x_7$ is the connection of the Lorentz metric $g$ and $F^#$ is the tensor field of type $(1,1)$ given by $F(X, Y) = g(F^#(X), Y)$. Furthermore,

$$y(0) = p, \quad y(\delta) = q.$$  

(2)

The main goal of the present paper is to find out under what conditions on the system $\Gamma(M, g, F, u)$ and the points $p, q \in M$ there exists a solution $y: [0, \delta] \rightarrow M$ of Eq. (1) with the boundary conditions (2) such that

$$\frac{g(dy/ds, dy/ds)}{2} + u(y) = 0.$$  

(3)

We refer to $\Gamma$ as a gyro system of relativistic type, and the solutions of system (1), (3) are called motions of $\Gamma$. If $\dim M = 4$, the form $F$ satisfies the Maxwell equations, and $u \equiv 1/2$, then system (1), (3) can be interpreted as the relativistic equations of motion of a charged particle in a gravitational and an electromagnetic field. Note that Eq. (3) is equivalent to the equation

$$D(\delta, x) = u(x)\delta^2 + \frac{1}{2}g(\dot{x}, \dot{x}) = 0.$$  

Thus, a curve $y: [0, \delta] \rightarrow M$ satisfies problem (1)-(3) if and only if $(\delta, x)$ is an extremal of $S$ and $D(\delta, x) = 0$.

Next, let $\nu: M^* \rightarrow M$ be the smooth universal covering and $\Phi = F/(\theta)$. Since the periods of the form $\nu^*\Phi$ over the 2-cycles of $M^*$ are trivial if $k = 0$ and form the group $\mathbb{Z}$ if $k = 1$, it follows that there exists a principal bundle $\eta = (\Xi, \xi, M^*, G)$ with base $M^*$, total space $\Xi$, projection $\xi: \Xi \rightarrow M^*$, structural group $G$, and characteristic class $[\nu^*\Phi] \in H^2(M^*, \mathbb{R})$. Set $\mu = \nu \circ \xi$ and $\Phi^* = \mu^* \Phi$. On $\Xi$ there exists a 1-form $\omega$ for which $d\omega = \Phi^*$ and a $G$-connection $H$ with connection form $\omega$ and curvature form $\Phi^*$. Furthermore, the formulas

$$ u^* = u \circ \mu, \quad \omega^* = \frac{(\theta)}{2u^*} \omega, \quad b = 2u^* \omega^* \otimes \omega^* + \mu^* g $$

define a Lorentz metric $b$ on $\Xi$. Suppose that $O$ is a continuous timelike vector field on $M$ and $O^*$ is the horizontal lift of $O$ with respect to $H$. Then $O^*$ is also continuous and $b(O^*, O^*) = g(O, O) < 0$. We assume that the Lorentz manifolds $(M, g)$ and $(\Xi, b)$ are oriented in time by the vector fields $O$ and $O^*$, respectively. For any $\delta \in \mathbb{R}$, $x \in \Omega_{pq}$, and $v \in \mu^{-1}(p)$ there exists a unique piecewise smooth path $\bar{x}: I \rightarrow \Xi$ such that $\mu \circ \bar{x} = x$, $\omega^*(d\bar{x}/d\tau) \equiv \delta$, and $\bar{x}(0) = v$. In this case we write $\bar{x} = \lambda^\nu(\delta, x)$ and say that $\bar{x}$ is the $\delta$-lift of the path $x$ [2].

We use the symbols $\Lambda_{pq}$, $\Lambda_{pp}$, and $\Lambda_{pp}^0$ to denote the subspaces of $\Omega_{pq}$ consisting of smooth paths on the manifold $(M, g)$ directed to the future and nonspacelike, timelike, or isotropic, respectively. If $v, w \in \Xi$, then $\Lambda_{vw}$, $\Lambda^{-}_{vw}$, and $\Lambda_{pw}^0$ stand for similar path spaces for the manifold $(\Xi, b)$.

**Lemma 1.** Let $J$ be an open or closed subinterval of $\mathbb{R}$, and let $\bar{x}: J \rightarrow \Xi$ be a nonspacelike curve. Then the curve $x = \mu \circ \bar{x}$ is also nonspacelike and is directed to the future or to the past simultaneously with $\bar{x}$.

**Proof.** First, assume that $\bar{x}$ is a smooth curve and set $\dot{x} = dx/d\tau$ and $\dot{\bar{x}} = d\bar{x}/d\tau$. By assumption,

$$2u(x)\omega^*(\dot{\bar{x}})^2 + g(\dot{x}, \dot{x}) = b(\bar{x}, \dot{\bar{x}}) \leq 0.$$  

Consequently, $g(\dot{x}, \dot{x}) \leq -2u(x)\omega^*(\dot{\bar{x}})^2 \leq 0$ and the curve $x$ is nonspacelike. The second assertion of the lemma follows from the identity $b(O^*, \bar{x}) = g(O, \dot{x})$. 

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