Theorem 4. Let \( r \in \mathbb{R}^n \), \( 0 < r_1 = \cdots = r_{l+1} < r_{l+2} \leq \cdots \leq r_n \), and \( 1 < p, q < \infty \). Then

i) for \( 1/p + 1/q < 1 \), \( p < 2 \), and \( r_1 > 1 - 1/q \) we have

\[
\alpha(N) \left( \frac{\log^l N}{N} \right)^{r_1 - 1/2 + 1/q} \log^{l/q} N \ll \lambda_N(H^r_p(T^n), L_q) \ll \left( \frac{\log^{l/q} N}{N} \right)^{r_1 - 1/2 + 1/q} \log^{l/q} N;
\]

ii) for \( 2 \leq p < q \) and \( r_1 > 1/p - 1/q \) we have

\[
\alpha(N) \left( \frac{\log^l N}{N} \right)^{r_1 - 1/p + 1/q} \log^{l/q} N \ll \lambda_N(H^r_p(T^n), L_q) \ll \left( \frac{\log^{l/q} N}{N} \right)^{r_1 - 1/p + 1/q} \log^{l/q} N,
\]

where \( \alpha(N) = (\log \log N)^{1/q - 1} \).

Results close to Theorem 4 (with the factor \( \alpha(N) \) replaced by \( (\log \log N)^{-1/\sqrt{\log \log N}} \)) were obtained by Galeev [4], who reduced the lower bound for \( \lambda_N(H^r_p, L_q) \) to that for the width \( \lambda_N(B_{\infty}^{m, \infty}, \ell_2^{m, \infty}) \), which, in turn, can easily be reduced to the estimate of the width \( d_N(B_{1, \infty}^{m, \infty}, \ell_2^{m, \infty}) \) in Theorem 1. However, Theorem 3 permits one to obtain the sharper result given in Theorem 4.

The author is grateful to B. S. Kashin for his permanent attention to this work and to É. M. Galeev for useful discussions.

References


STEKLOV MATHEMATICS INSTITUTE, RUSSIAN ACADEMY OF SCIENCES
E-mail address: izaak@ipsun.ac.msk.su

Translated by V. E. Nazaikinskii

---

Mathematical Notes, Vol. 59, No. 3, 1996

On the Interpolation Constants of Whitney

N. V. Kryakin and M. D. Takev

§1. For a function \( f(x) \) defined on \( I = [0, 1] \), we put

\[
\Delta_n^h f(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x + jh).
\]

The main result of this note is

Original article submitted June 20, 1994.

0001-4346/96/5934-0330$15.00 ©1996 Plenum Publishing Corporation
Theorem 1. Let \( f(x) \in C(I) \) and \( Q_{n-1}(f; x) \) be an algebraic polynomial of degree not higher than \( n-1 \) such that

\[
f\left(\frac{i}{n-1}\right) = Q_{n-1}\left(f; \frac{i}{n-1}\right), \quad i = 0, \ldots, n-1.
\]

Then

\[
\sup_{z \in I} |f(x) - Q_{n-1}(f; x)| \leq W'(n) \sup_{y, y + nh \in I} \left|\Delta^n_y f(y)\right|, \quad \text{with } W'(n) < 5.
\]

Theorem 1 refines the estimate of the Whitney interpolational constants \( W'(n) < 38 \) obtained in [1]. Note that according to B. Sendov's conjecture [2], \( W'(n) < 2. \)

§2. We shall need the following lemma.

Lemma A [3]. Let \( f \in C(I), n \in \mathbb{N} \) and \( \int_0^{1/n} f(t) \, dt = 0, i = 1, \ldots, n. \) Then for \( x \in [1/(n+1), 1/n], \)

\[
f(ix) = \varphi_i(x) - \int_x^{1/n} \sum_{j=1}^n i^{j-1} \varphi_j(y) \left(l_{n,j}\left(\frac{iy}{x}\right)\right) \, dy,
\]

where

\[
\varphi_i(x) = (-1)^i \binom{n}{i}^{-1} \frac{1}{x} \int_x^{1/n} \Delta^n_y f(i(x - y)) \, dy, \quad l_{n,j}(t) = \prod_{k \neq j}^{n-1} \frac{t - k}{j - k}, \quad j = 1, \ldots, n.
\]

§3. Sketch of the proof. Assume that

\[
\sup_{y, h} \left|\Delta^n_y f(y)\right| \leq 1.
\]

We want to prove that

\[
\left|f(x) - Q_{n-1}(f; x)\right| < 5.
\]

We shall do this indirectly, using "interpolational in the mean" polynomials. They are defined by the following conditions:

\[
\int_0^{1/n} (f(x) - P_{n-1}(f; x)) \, dx = 0, \quad i = 1, \ldots, n.
\]

We have

\[
\left|f(x) - Q_{n-1}(f; x)\right| \leq \left|f(x) - P_{n-1}(f; x) - Q_{n-1}(f; x) + P_{n-1}(f; x)\right|
\]

\[
\leq \left|f(x) - P_{n-1}(f; x)\right| + \left|Q_{n-1}(f - P_{n-1}; x)\right|.
\]

The estimate for the first summand is given in [4]

\[
\left|f(x) - P_{n-1}(f; x)\right| \leq 2.
\]

The estimate of the second summand

\[
\left|Q_{n-1}(f - P_{n-1}; x)\right| = \left|\sum_{i=0}^{n-1} \tilde{f}(q_i) l_{n-1,i}(x(n-1))\right|
\]

where

\[
q_i = \frac{i}{n-1}, \quad \tilde{f}(x) = f(x) - P_{n-1}(f; x),
\]

is based on Lemma A.

The application of this lemma to \( \tilde{f}(x) \) and accurate calculation of the resulting sums leads to the following inequality:

\[
\left|\sum_{i=0}^{n-1} \tilde{f}(q_i) l_{n-1,i}(x(n-1))\right| < 3.
\]

Thus,

\[
\left|f(x) - Q_{n-1}(f; x)\right| < 2 + 3 = 5.
\]