The use of polynomials interpolating in the mean as an intermediate approximation yields better results if we modify the uniform point interpolation. Specifically, the following theorem holds.

Theorem 2. Let \( f(x) \in C(I) \) and \( R_{n-1}(f; x) \) be the algebraic polynomial of degree not higher than \( n-1 \) \( (n \geq 2) \) such that

\[
\text{for } n = 2k \quad f\left(\frac{i}{n}\right) = R_{n-1}\left(f; \frac{i}{n}\right), \quad i = 0, \ldots, n, \quad i \neq k;
\]

\[
\text{for } n = 2k + 1 \quad f\left(\frac{i}{n}\right) = R_{n-1}\left(f; \frac{i}{n}\right), \quad i = 0, \ldots, n, \quad i \neq k, k + 1;
\]

\[
f\left(\frac{1}{2}\right) = R_{n-1}\left(f; \frac{1}{2}\right).
\]

Then

\[
\sup_{x \in I} |f(x) - R_{n-1}(f; x)| \leq 3 \sup_{y, y+nh \in I} |\Delta_h^n f(y)|.
\]

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On Connectedness and Discontinuity of Invariant Sets

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§1. Definitions and statements

A set \( M \subset X \), where \((X, d)\) is a complete metric space, is called invariant with respect to a collection of contraction mappings \( L = \{s_1, \ldots, s_N\} \) acting on \( X \) if

\[
M = \bigcup_{i=1}^{N} s_i M.
\]

Hutchinson [1] proved that for any collection of contraction mappings \( L = \{s_1, \ldots, s_N\} \) in a complete metric space there exists a unique compact subset invariant with respect to this collection. We denote
such a set depending on the collection $L$ by $|L|$. In what follows, we shall make use of the Lipschitz constant for a mapping $s : X \to X$

$$\text{Lip } s = \sup_{z \neq y} \frac{d(s(x), s(y))}{d(x, y)}.$$ 

**Definition.** The invariant set $|L|$ is called *totally discontinuous* if $s_i |L| \cap s_j |L| = \emptyset$ for $i \neq j$.

**Theorem 1.** Let $c_1, \ldots, c_N \in X$ be a set of $N$ different points. Then there exists a number $k$, $0 < k < 1$, such that for any collection $L = \{s_1, \ldots, s_N\}$ of contraction mappings, where $c_i$ is the fixed point for $s_i$ and $\max_i \text{Lip } s_i < k$, the invariant set $|L|$ is totally discontinuous.

**Theorem 2.** Let $L = \{s_1, \ldots, s_N\}$ be a collection of nondegenerate linear contraction mappings acting in Euclidean space $\mathbb{R}^n$. Then the following statements are valid:

1) If $\min_i \det s_i > N^{-1}$, then the invariant set $|L|$ is not totally discontinuous.

2) If $\min_i \det s_i > 2^{-1}$, then the invariant set $|L|$ is arcwise connected.

\section{2. Proofs of the theorems}

**Lemma 1.** Let $N$ points $c_1, \ldots, c_N \in X$ be given. Then for any $k$, $0 < k < 1$, there exists a closed ball $B_k$ containing any invariant set $|L|$, $L = \{s_1, \ldots, s_N\}$ under the condition that $c_i$ is the fixed point of $s_i$, for each $i$ and $\max_i \text{Lip } s_i < k$.

**Proof.** Let $B$ be a ball of radius $r$ containing all the points $c_i$. The ball $B_k$ of radius $R(k) = 2r/(1-k)$ concentric to the ball $B$ is the required one. It is easy to verify that

$$\bigcup_{i=1}^N s_i B_k \subset B_k,$$

and, therefore, $|L| \subset B_k$. \hfill $\square$

**Proof of Theorem 1.** Let $N$ points $c_1, \ldots, c_N \in X$ be given, and fix a number $k_0$, $0 < k_0 < 1$. Using Lemma 1, let us consider the ball $B_{k_0}$ of radius $R$, $d = \min_{i \neq j} d(c_i, c_j)$, $k = \min \{d/(2R), k_0\}$.

Let $L = \{s_1, \ldots, s_N\}$ be an arbitrary collection of contraction mappings such that $s_i c_i = c_i$ and $\text{Lip } s_i < k$, $k = 1, \ldots, N$. The triangle inequality implies that, for $i \neq j$, $s_i B_{k_0} \cap s_j B_{k_0} = \emptyset$. Further, by Lemma 1 we have $|L| \subset B_{k_0}$. The last two expressions show that the invariant set $|L|$ is totally discontinuous. Theorem 1 is proved. \hfill $\square$

**Definition (Hutchinson).** A collection $L = \{s_1, \ldots, s_N\}$ satisfies the open set condition if there exists a nonempty open set $O$ such that

1) $\bigcup_{i=1}^N s_i O \subset O$;  
2) $s_i O \cap s_j O = \emptyset, \quad i \neq j$.

**Lemma 2 (Hutchinson).** If $|L|$ is totally discontinuous, then $L$ satisfies the open set condition.

Let us show how this lemma implies statement 1) of Theorem 2. Suppose that $\min_i \det s_i > N^{-1}$, but the invariant set $|L|$ is totally discontinuous. Then $L$ satisfies the open set condition with respect to some open set $O$. Denote the volume of this set by $\text{Vol } O$. Since the contraction mappings from $L$ are nondegenerate linear transformations, we obtain

$$\text{Vol } O \geq \sum_{i=1}^N \text{Vol } s_i O = \text{Vol } O \sum_{i=1}^N \det s_i;$$

dTherefore, $\sum_{i=1}^N \det s_i \leq 1$. The last relation contradicts the assumption in 1) of Theorem 2, which requires that $\min_i \det s_i > N^{-1}$.

Now let us prove statement 2) of Theorem 2.